# Spherical Centroidal Voronoi Tessellations: Theory, Analysis, and Practical Issues
# Part II

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http://public.lanl.gov/ringler/ringler.html

# Outline

- Brief overview of Motivation
- Brief definition of SCVTs
- Potential finite-volume grid staggerings
- Challenges related to SCVTs
- Some collocated (A-grid) results
- SCVT testbed
- Conclusions

# Motivation

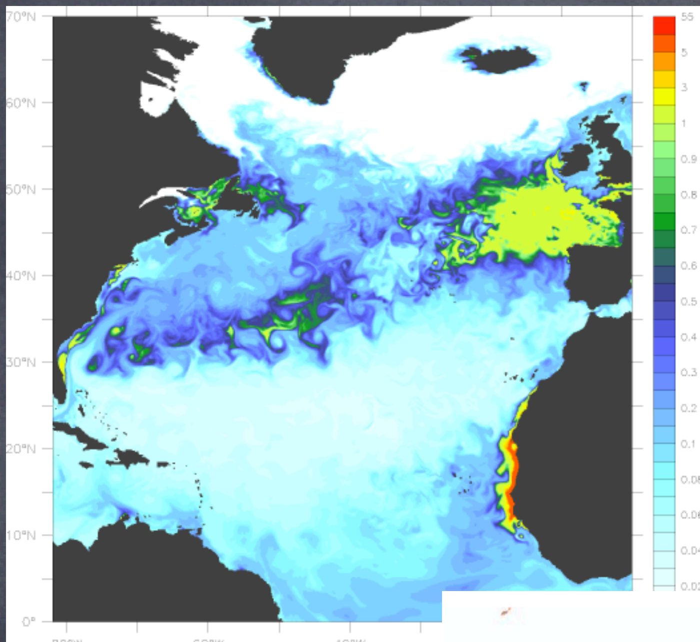
Mission of Department of Energy Climate Change Research Division: 'deliver improved scientific data and models about the potential response of the Earth's climate to increased greenhouse gas levels for policy makers to determine safe levels of greenhouse gases in the atmosphere.'

Process exist that impact the 'potential response of the Earth's climate to increased greenhouse gas levels' yet have spatial scales far below the current resolution of IPCC-class climate models.

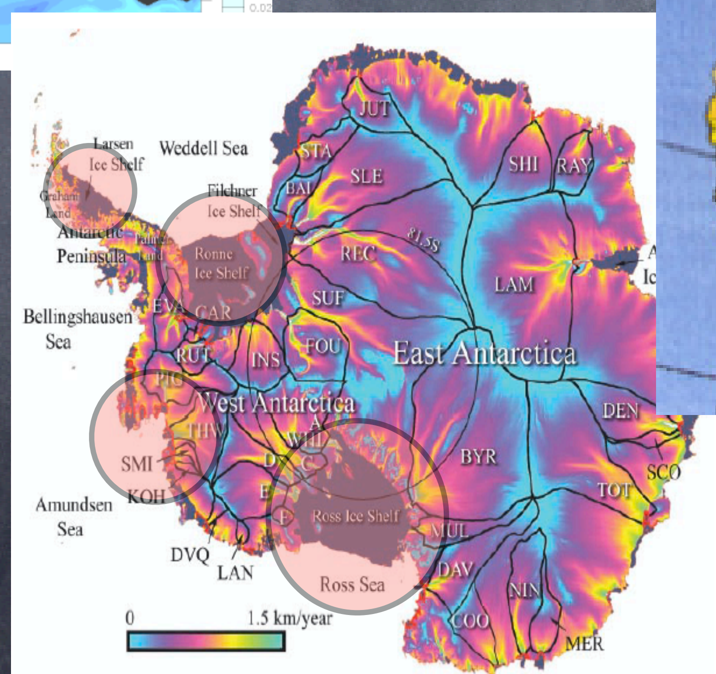
How do we deal with these processes, yet retain a computationally-tractable model of the Earth's climate?

We propose to do this through the application of variable-resolution, unstructured meshes. Target Earth system processes include oceans, ice sheets, ice shelves, biogeochemistry, and beyond.

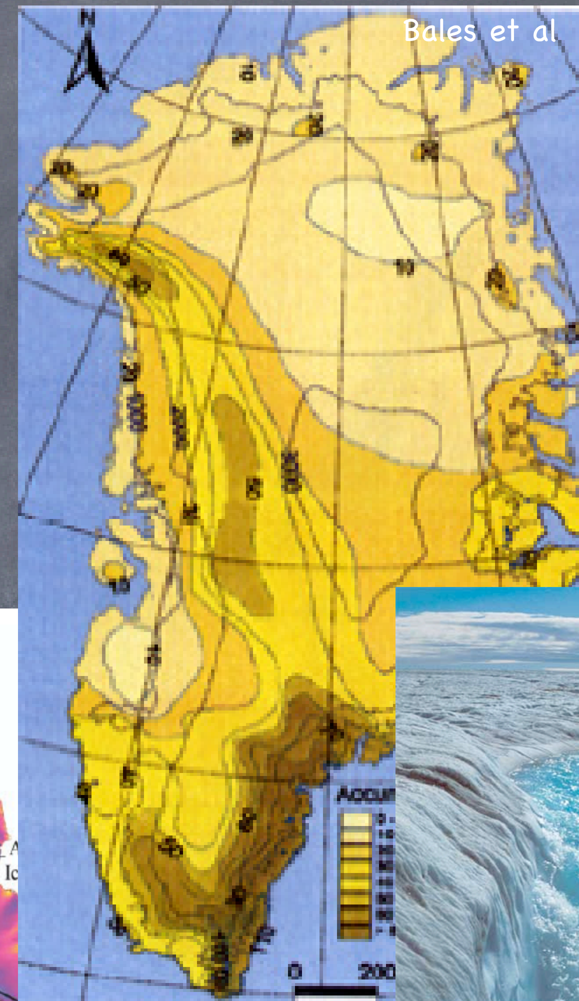
# Examples of important multi-scale processes



ocean biogeochemistry



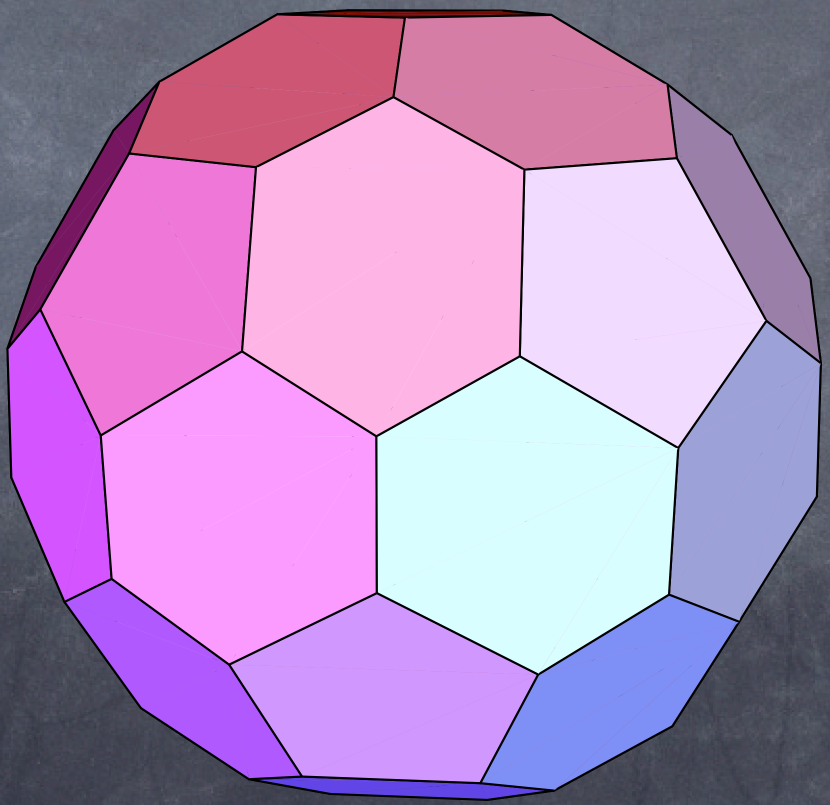

ice shelf-ocean interaction





sea-level rise

We propose to accommodate the requirement to resolve multiple scales within a single model by using Spherical Centroidal Voronoi Tessellations.

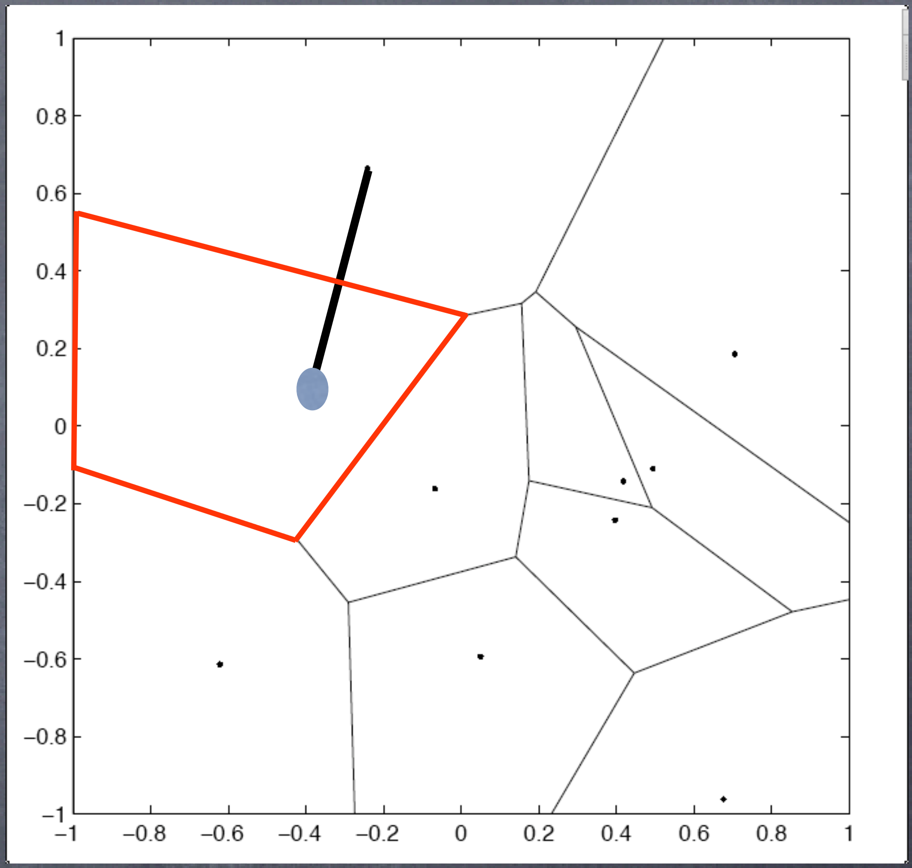


# Definition of a Voronoi Tessellations

Given a region, S
And a set of generators, $z_i$ ...

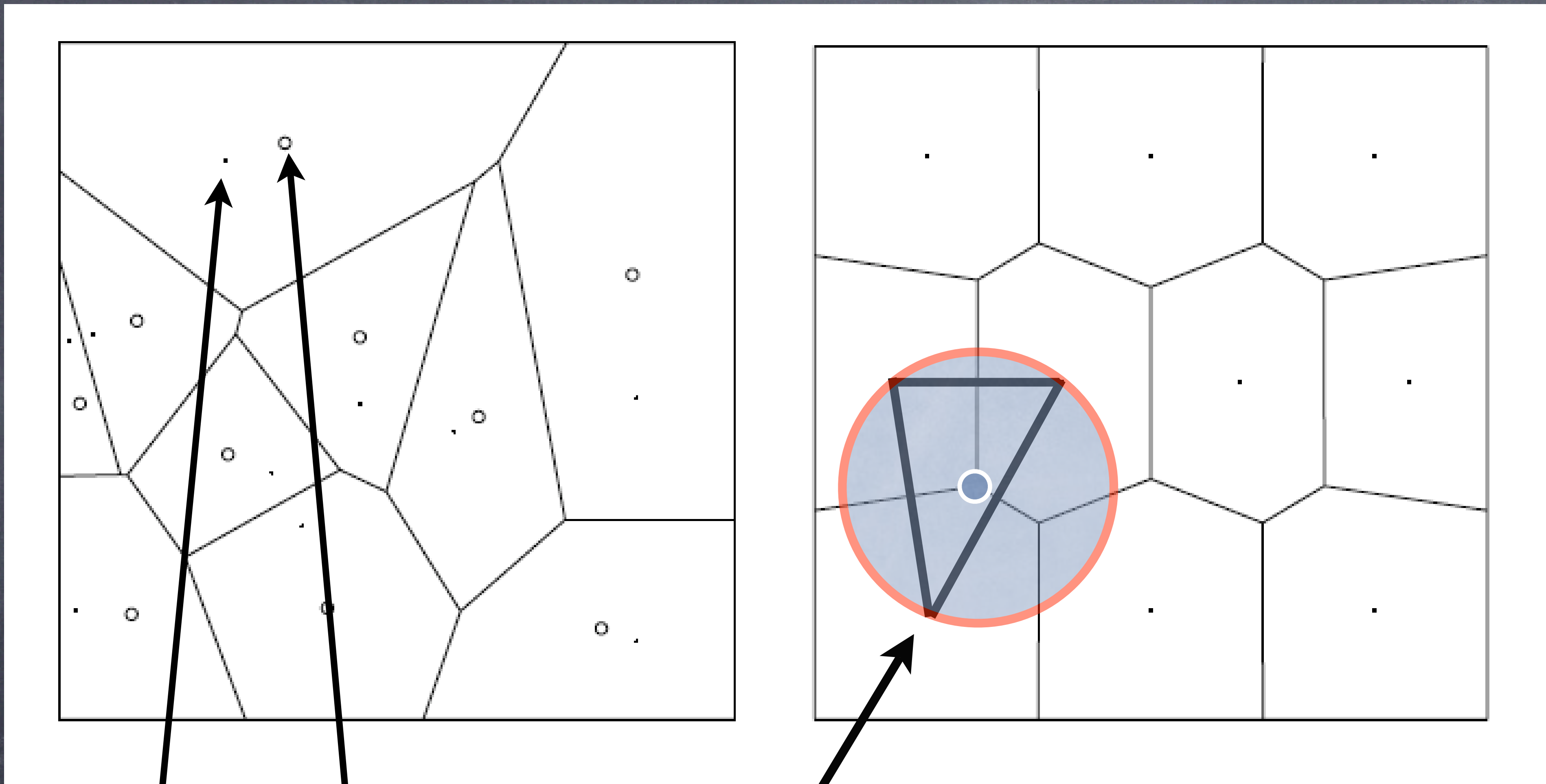
The Voronoi region, $V_i$, for each $z_i$ is the set of all points closer to $z_i$ than $z_j$ for j not equal to i.

We are guaranteed that the line connecting generators is orthogonal to the shared edge and is bisected by that edge.




But this does not mean that the grid is nice ....

# Definition of a Centroidal Voronoi Tessellations



Dual tessellation vertex at circumcenter

$z_i$

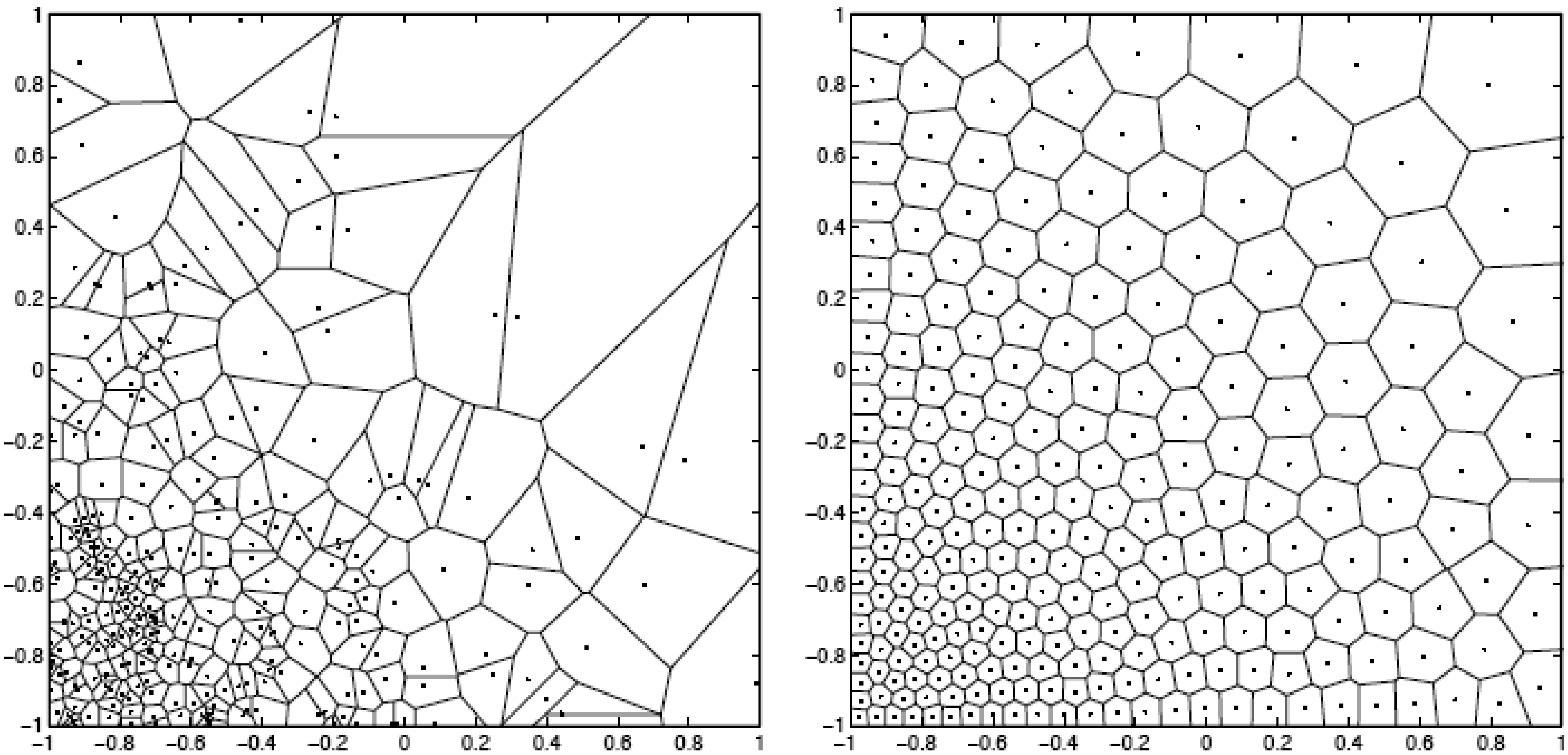
$z_i^*$ = center of mass wrt a user-defined density function

$$z^* = \frac{\int_V w\rho(w)\,dw}{\int_V \rho(w)\,dw}$$

# Non-uniform Centroidal Voronoi Tessellations

Distribute generators in such a way as to make the grid regular.

Also biases the location of those generators to regions of high density.




Random sampling

Centroidal Voronoi

# (S)CVTs have their roots in applied math ...

Gersho conjecture (now proven in 2D): as we added generators, all cells evolve toward perfect hexagons. Meaning that the grid just keeps getting more regular as we add resolution.

Optimal sampling: given a region, R, and N buckets to measure precipitation in R, the optimal placement of those buckets is a CVT. If a prior distribution, P, of precipitation is known, the CVT takes that information into account with rho=P^1/2.

Guaranteed to have 2nd-order truncation error of Poisson equation.

In summary: if Voronoi tessellations are to be used, then there is no good reason not to use Centroidal Voronoi Tessellations.

A mesh is no better than the numerical method we use with it to discretize the continuous equations.

So what method should we use for variable resolution SCVTs?

# A scoping of potential methods for SCVTs

| Method | geostrophic adjustment | vortex dynamics | energy conservation | computational modes | computational efficiency | maps to unstructured Voronoi diagram |
|---|---|---|---|---|---|---|
| ZM-grid (staggered FV) | + | + | + | - | . | -- |
| A-grid (collocated FV) | - | -- | + | + | + | ++ |
| Z-grid (collocated vor-div FV) | + | + | . | ++ | -- | + |
| Hex C-grid (proj FV based on VD) | -- | . | . | -- | . | + |
| Tri C-grid (proj FV based on DT) | + | + | + | - | . | + |

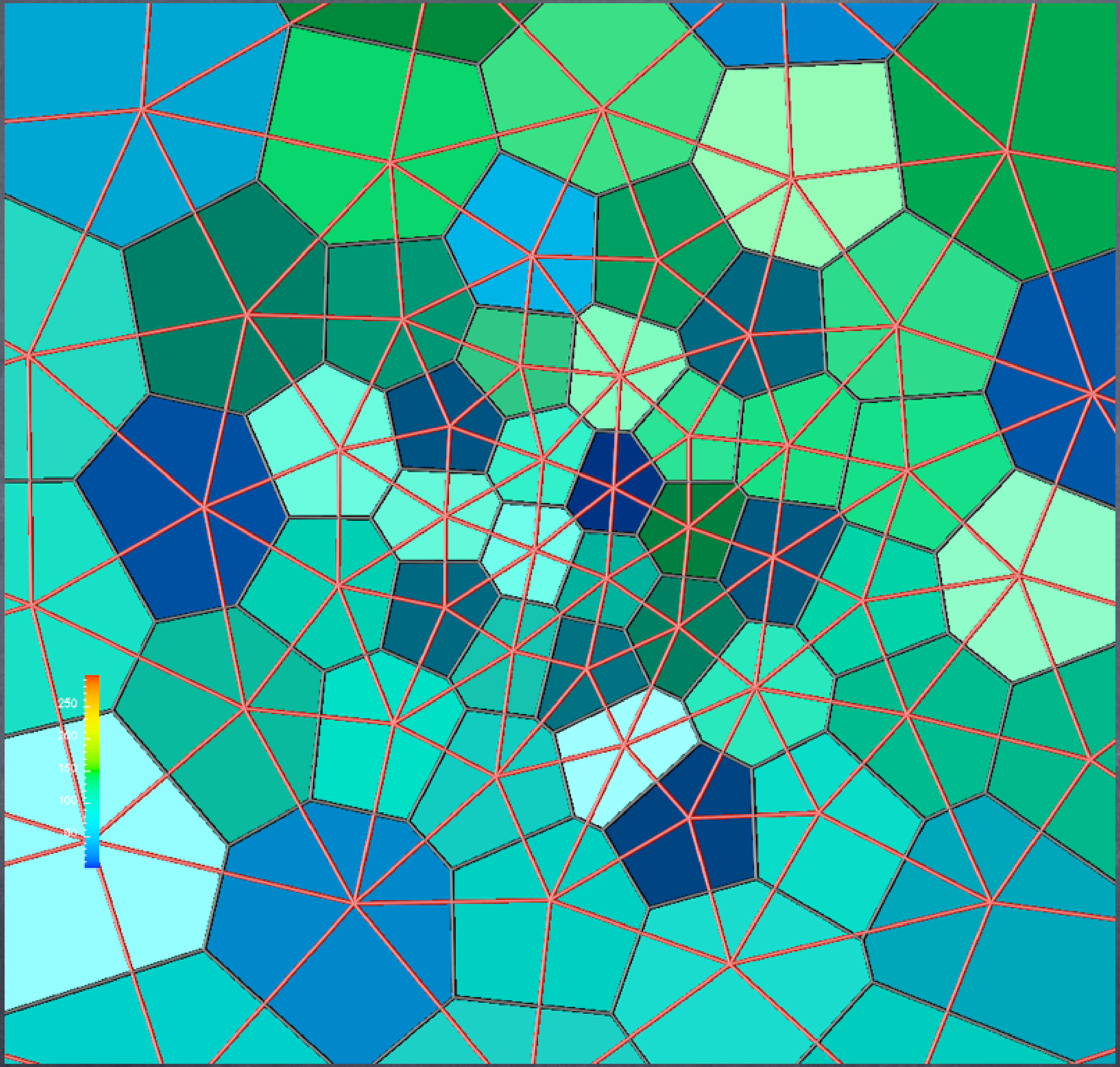
FV: Finite Volume
VD: Voronoi Diagram
DT: Delaunay Triagulation

\+ : positive aspect
\- : negative aspect
++ : outstanding relative to others
-- : deficiency of concern
. : neutral

I will use the ZM-grid to highlight the challenges associated with variable-resolution Voronoi tessellations.

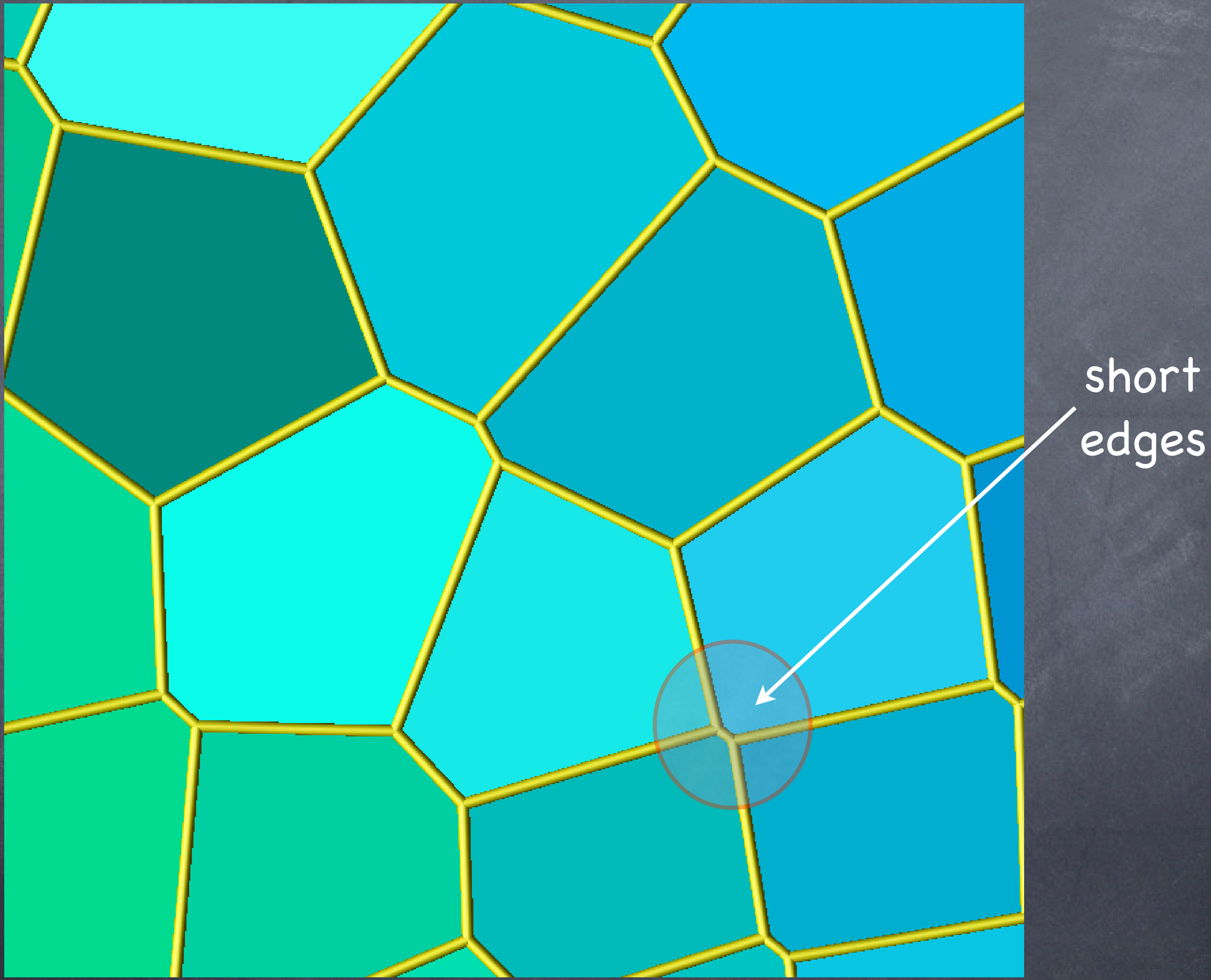
Note: the grids we will look at are designed to be beyond worst-case scenario.

# Variable Resolution SCVTs come with their own issues.



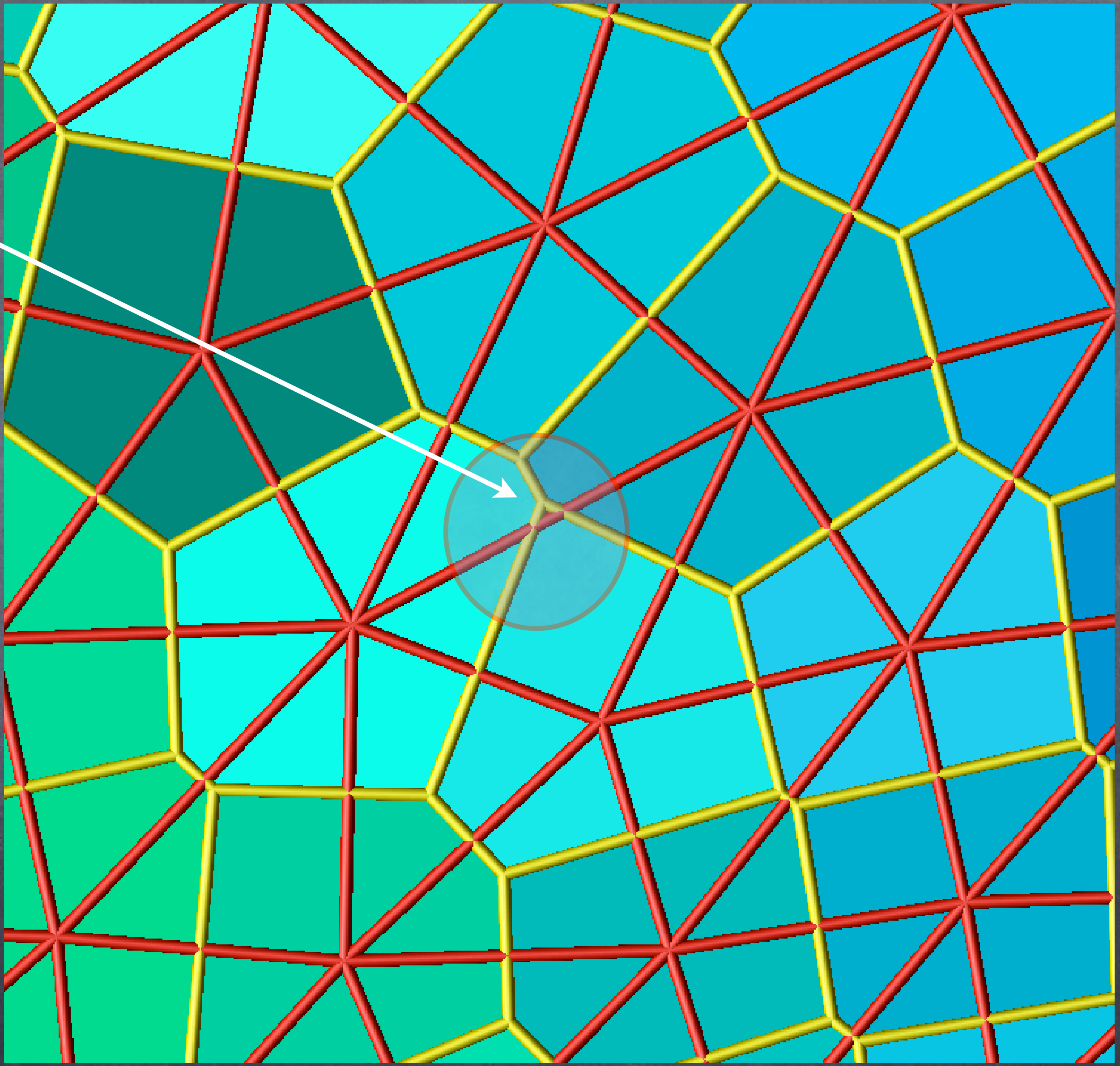
Randomly color Voronoi Diagram and Delaunay Triangulation

# A closer look

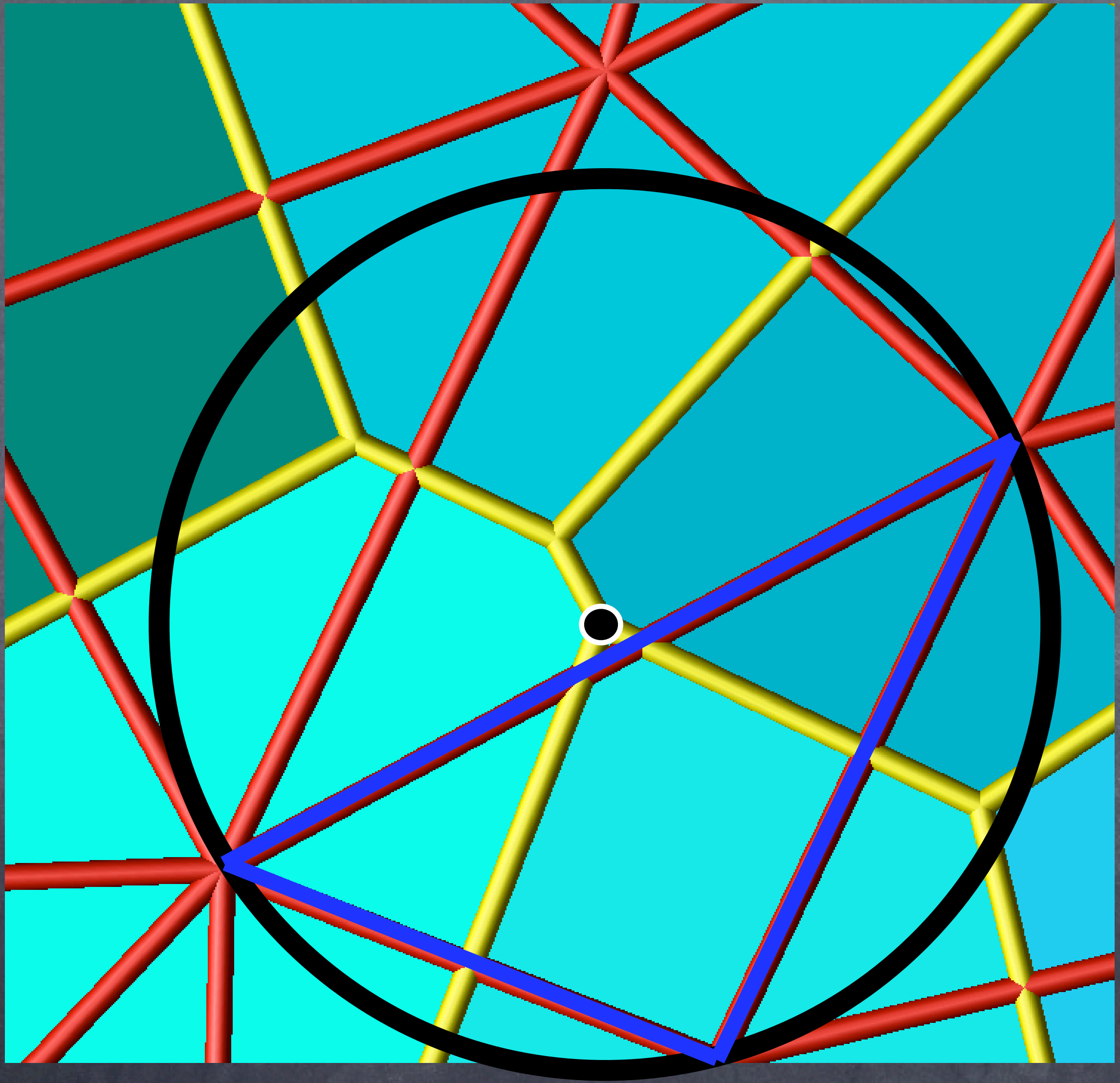


# Adding the Delaunay triagulation ....

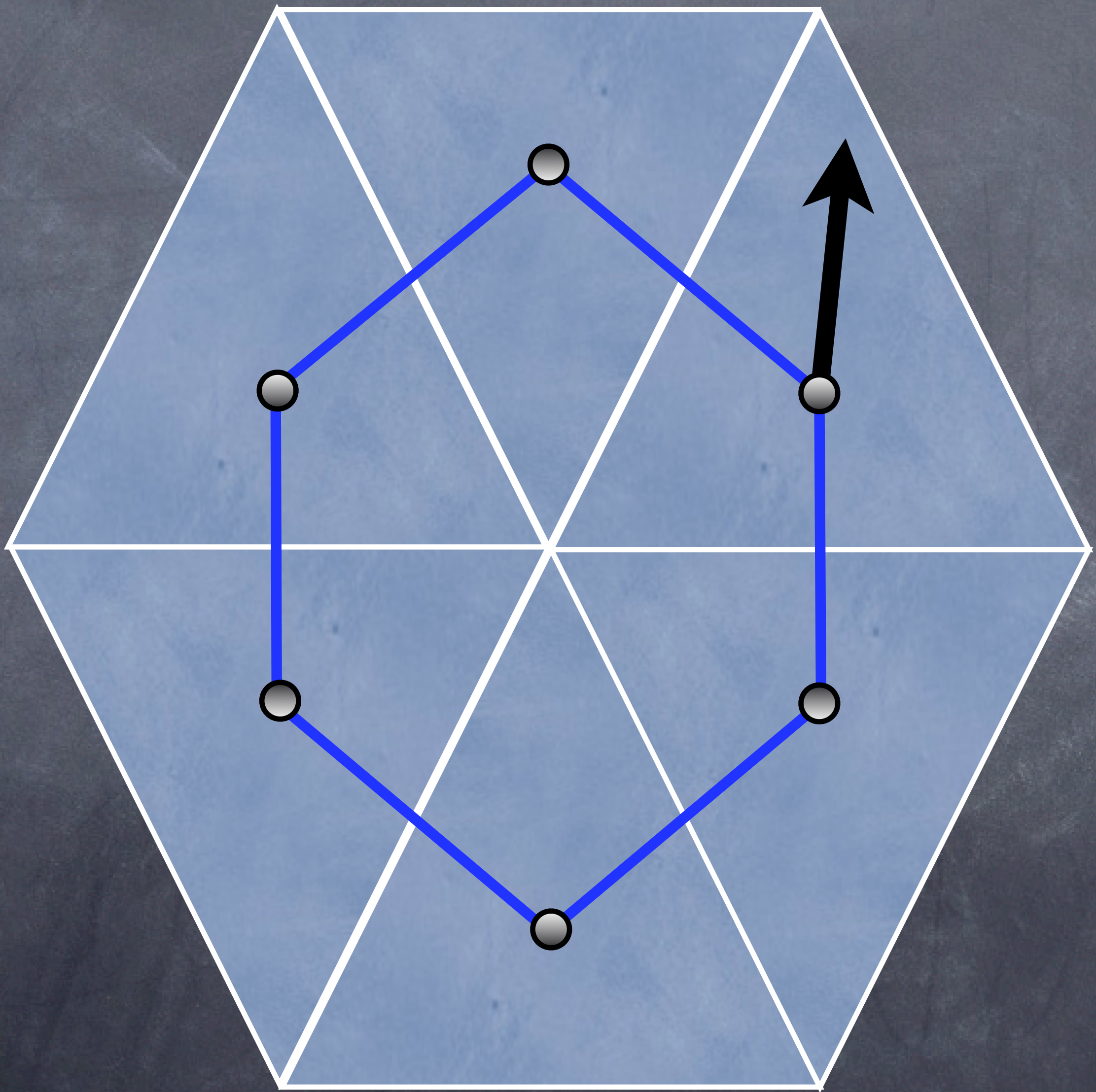
circumcenter
falls outside
triangle



# Delaunay triangulation and Circumcenter
one-to-one correspondence between triangles and vertices

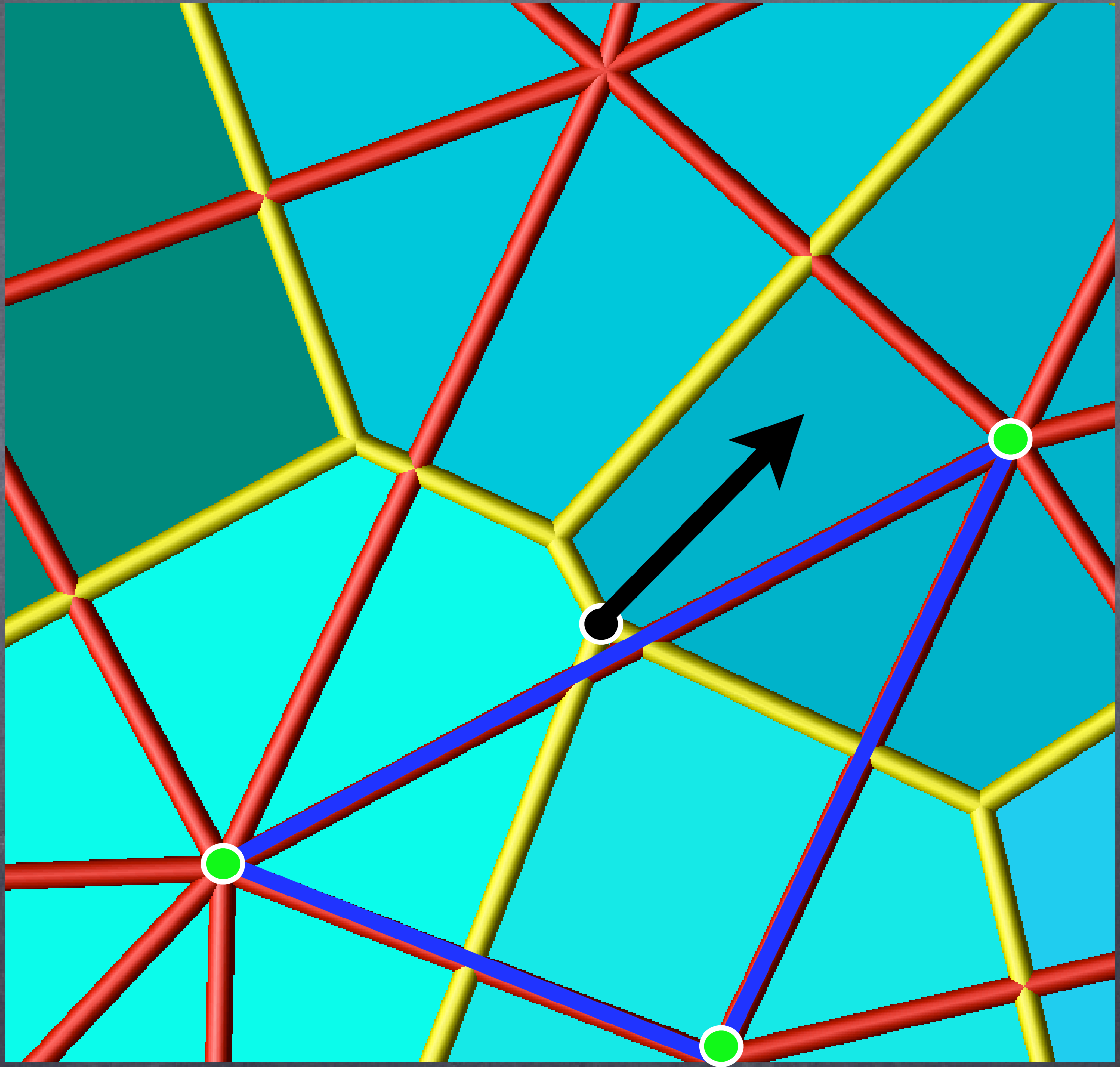


# The ZM-grid staggering is not a natural fit for this situation.



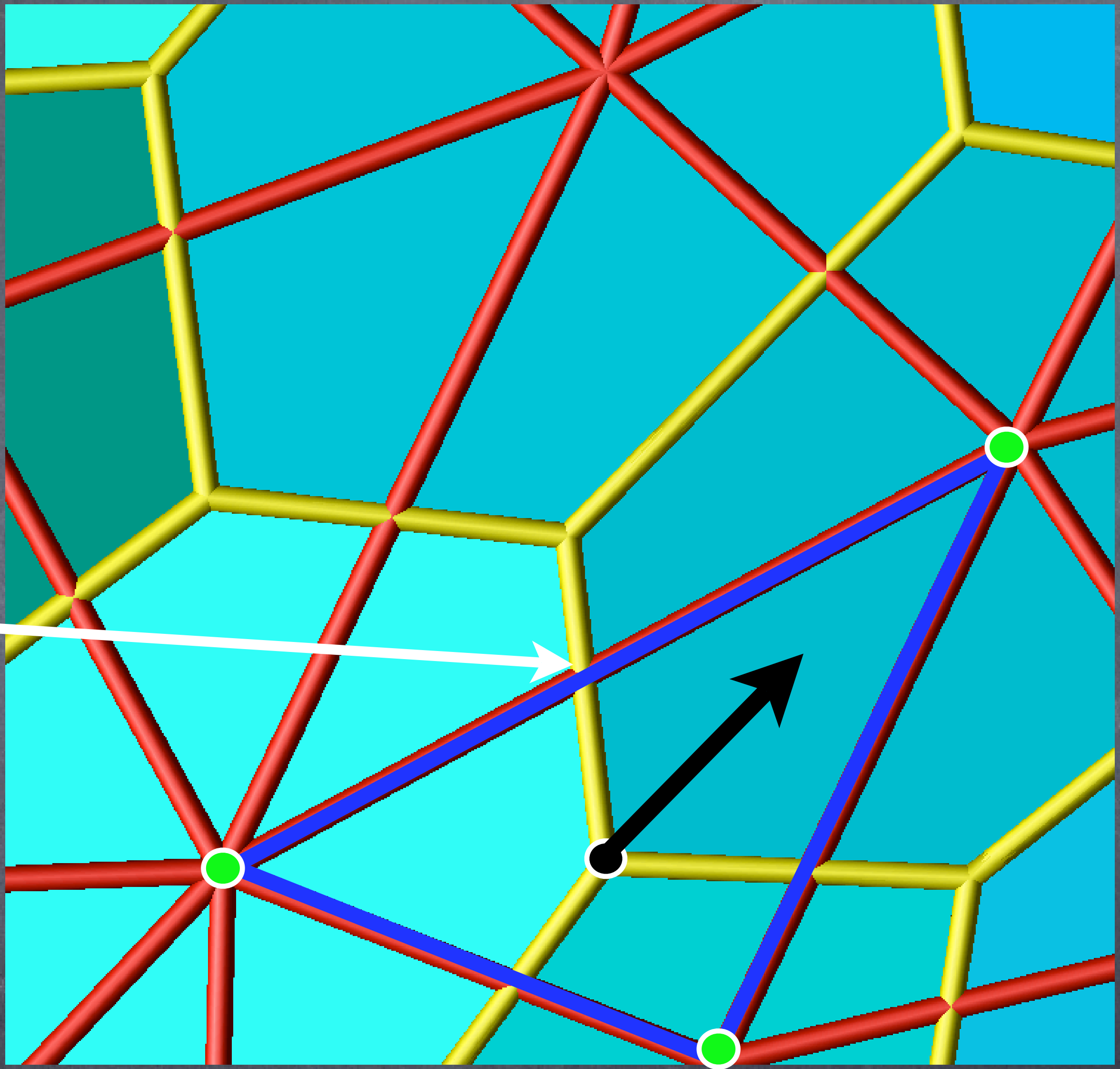
When we use the ZM staggering, we do not want the velocity point to fall outside its corresponding triangle.

- mass points
- velocity points



# The barycenter (or similar) alternative ...

The generators have not moved, but we now place the vertices at the barycenter.



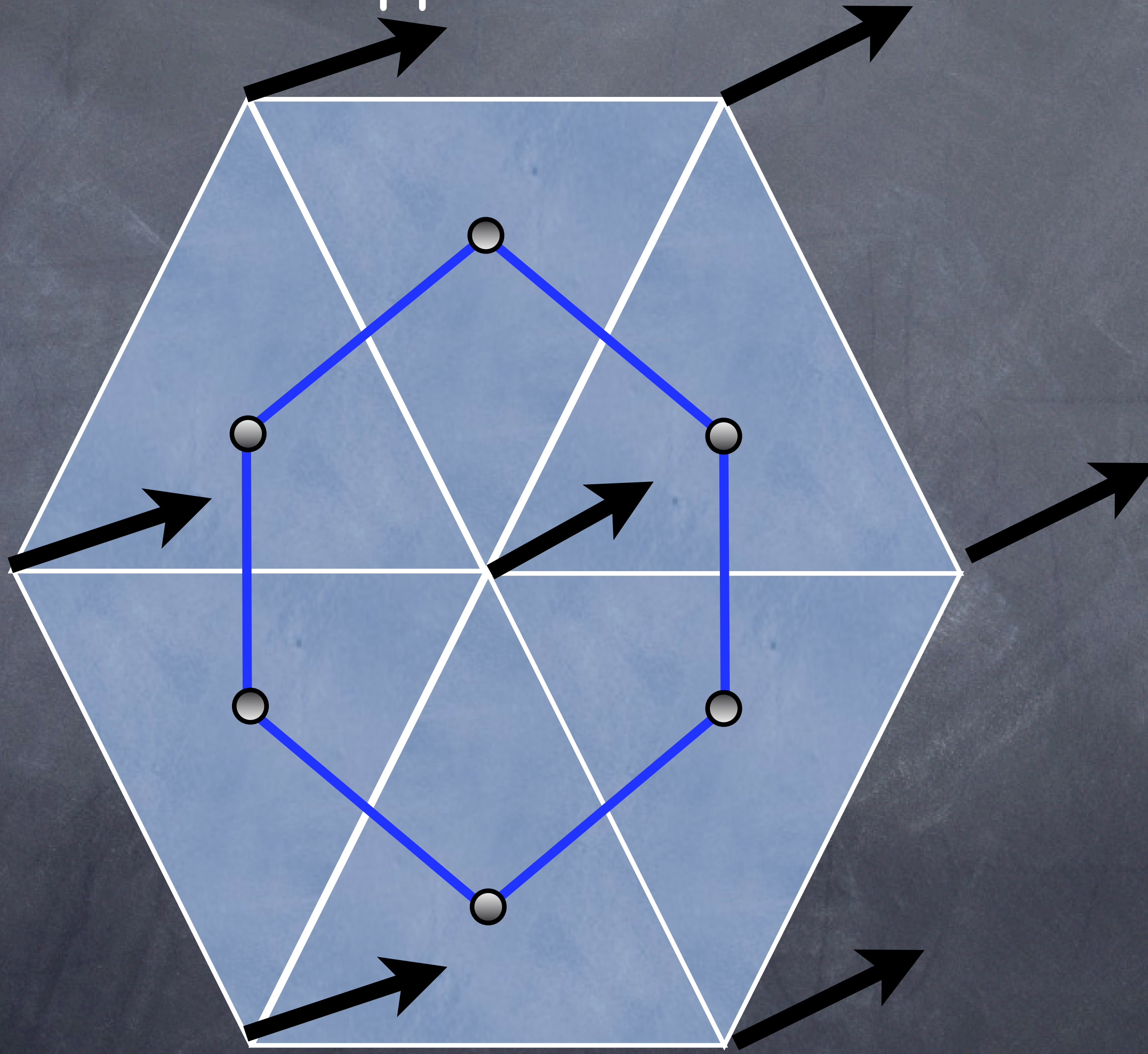
Intersection is no longer orthogonal

Unfortunately we lose the properties of the Voronoi Diagriam and the 100 years of mathematical analysis built on Voronoi diagrams.

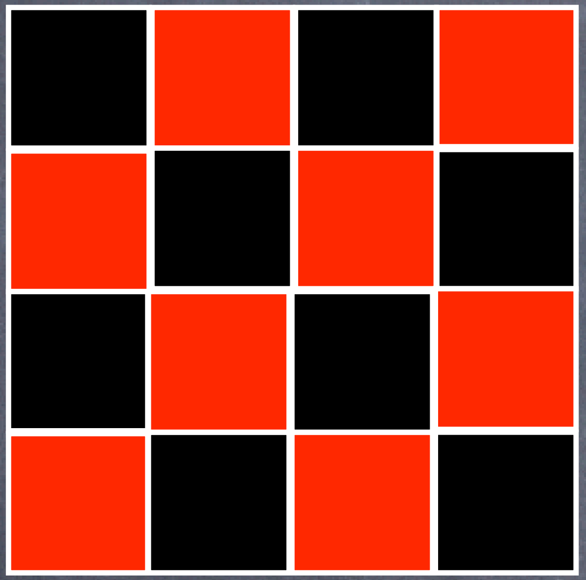
The preceding discussion is meant to highlight the challenges of using variable resolution SCVT grids.

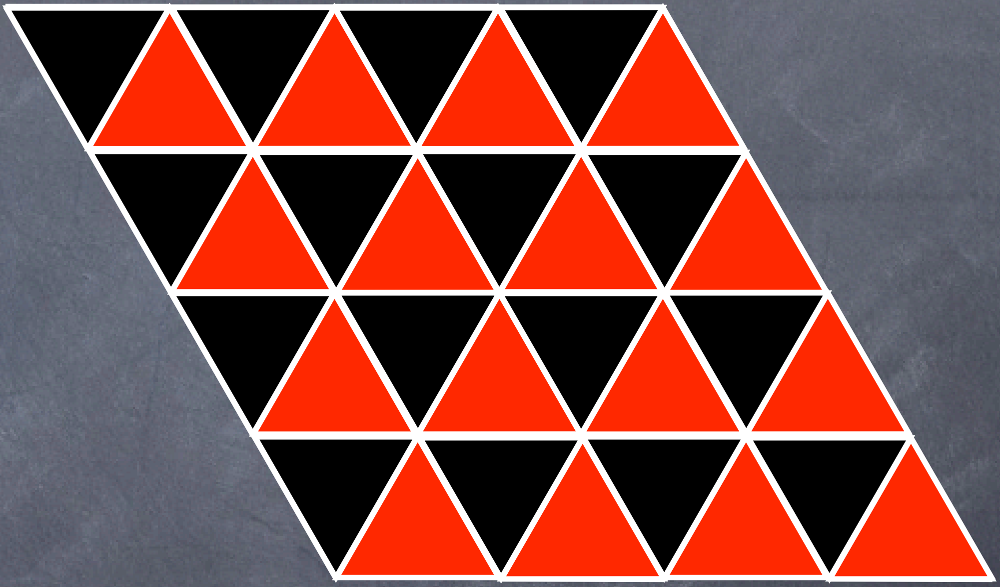
The ZM-grid may turn out to be the best choice, but it is useful here to explore other choices.

# Turning to the collocated (A-grid) FV approach



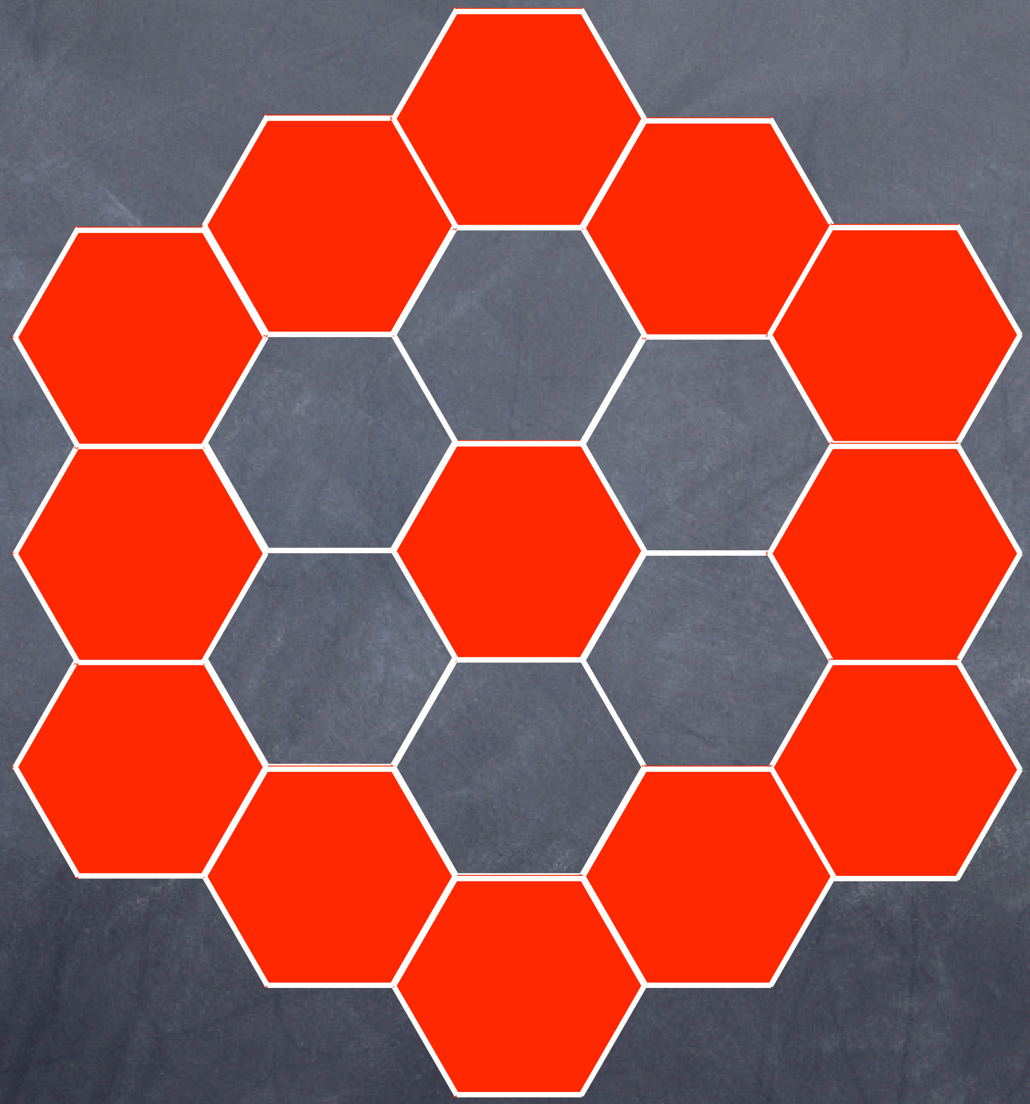
# The null-space of collocated quad and triangle grid is not representative of the "hexagonal" collocated grid.





Null space of Laplacian contains the red-black checkerboard.

The Laplacian is not nearly as compact as the Z-grid, ZM-grid, or Hex C-grid --- but there is no checkerboard pattern.



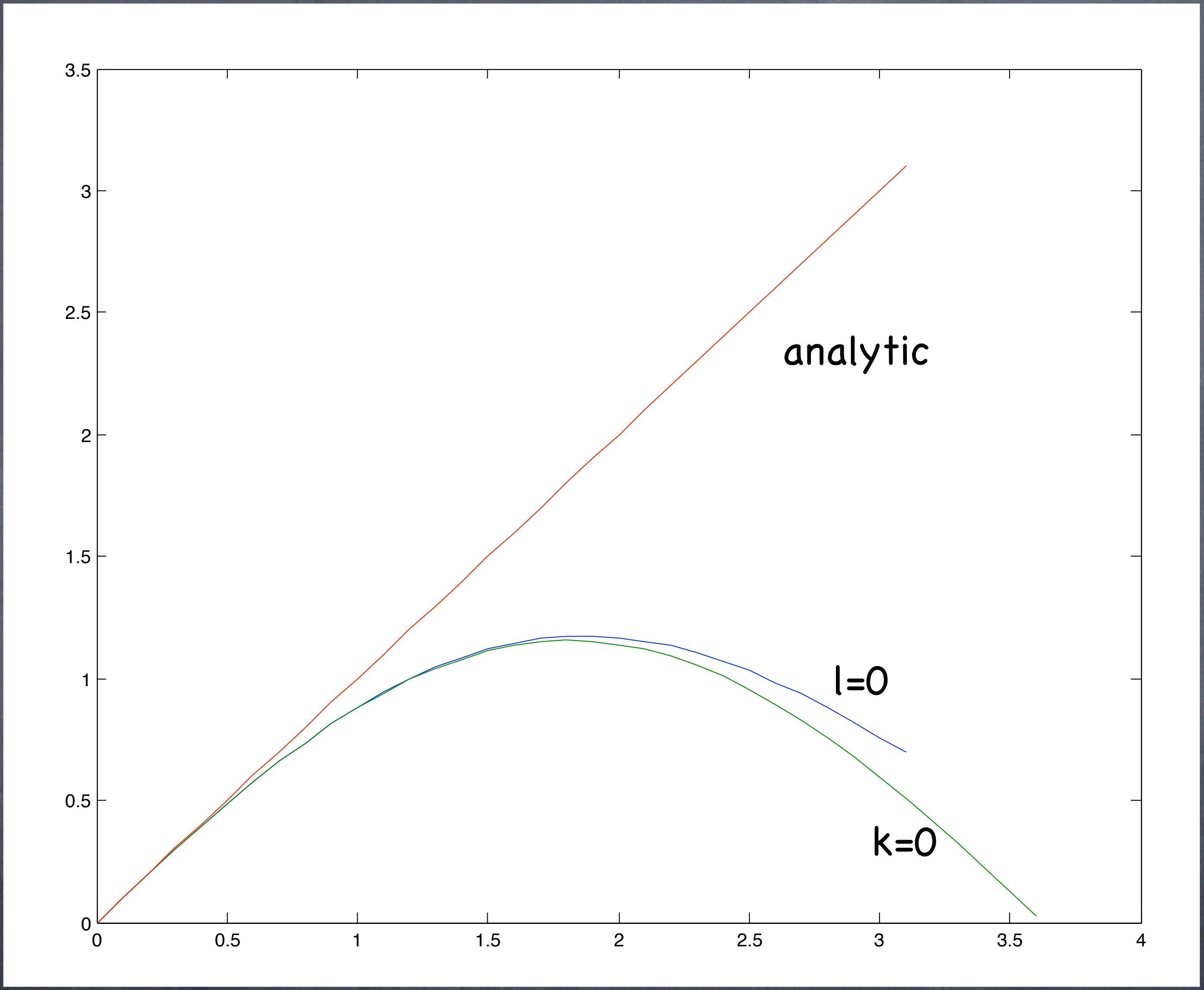
# Dispersion relation for collocated hexagonal grid

$$\left(\frac{\omega}{f}\right)^2 = 1 + \left(\frac{\lambda}{d}\right)^2 \left[ Y^2 + \left(\frac{X^2}{3}\right) \right]$$

$$Y = \sin\left( +\frac{kd}{2} - \frac{\sqrt{3}}{2} ld \right) + \sin\left( -\frac{kd}{2} - \frac{\sqrt{3}}{2} ld \right)$$

$$X = 2 * \sin(kd) + \sin\left( +\frac{kd}{2} - \frac{\sqrt{3}}{2} ld \right) - \sin\left( -\frac{kd}{2} - \frac{\sqrt{3}}{2} ld \right)$$

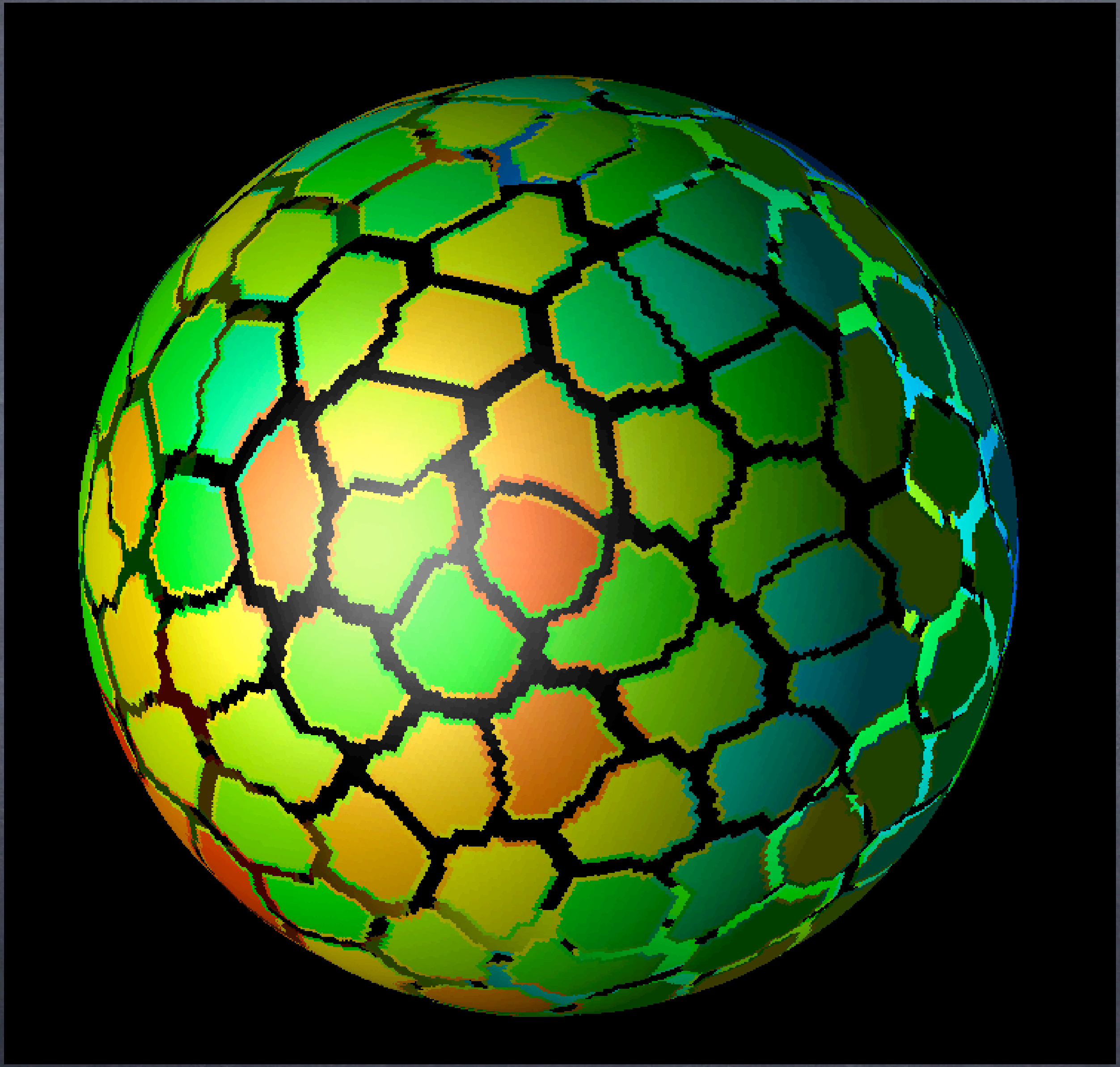
# Dispersion along k=0 and l=0




While there is no checkerboard pattern, there is a possibility for "striping."

As a part of this scoping process, we have written a shallow-water model based on collocation.

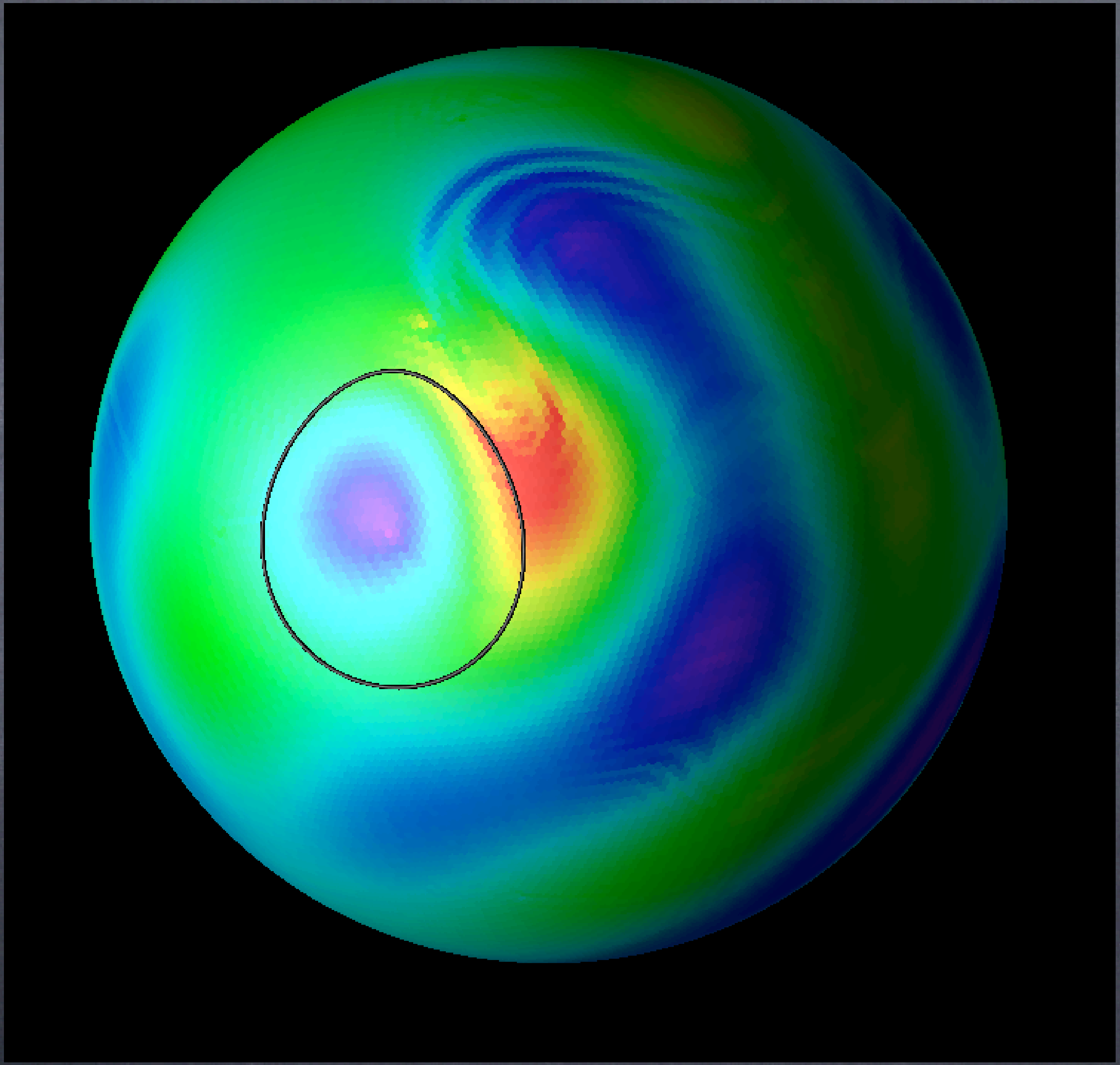
# Designed for high-performance computing



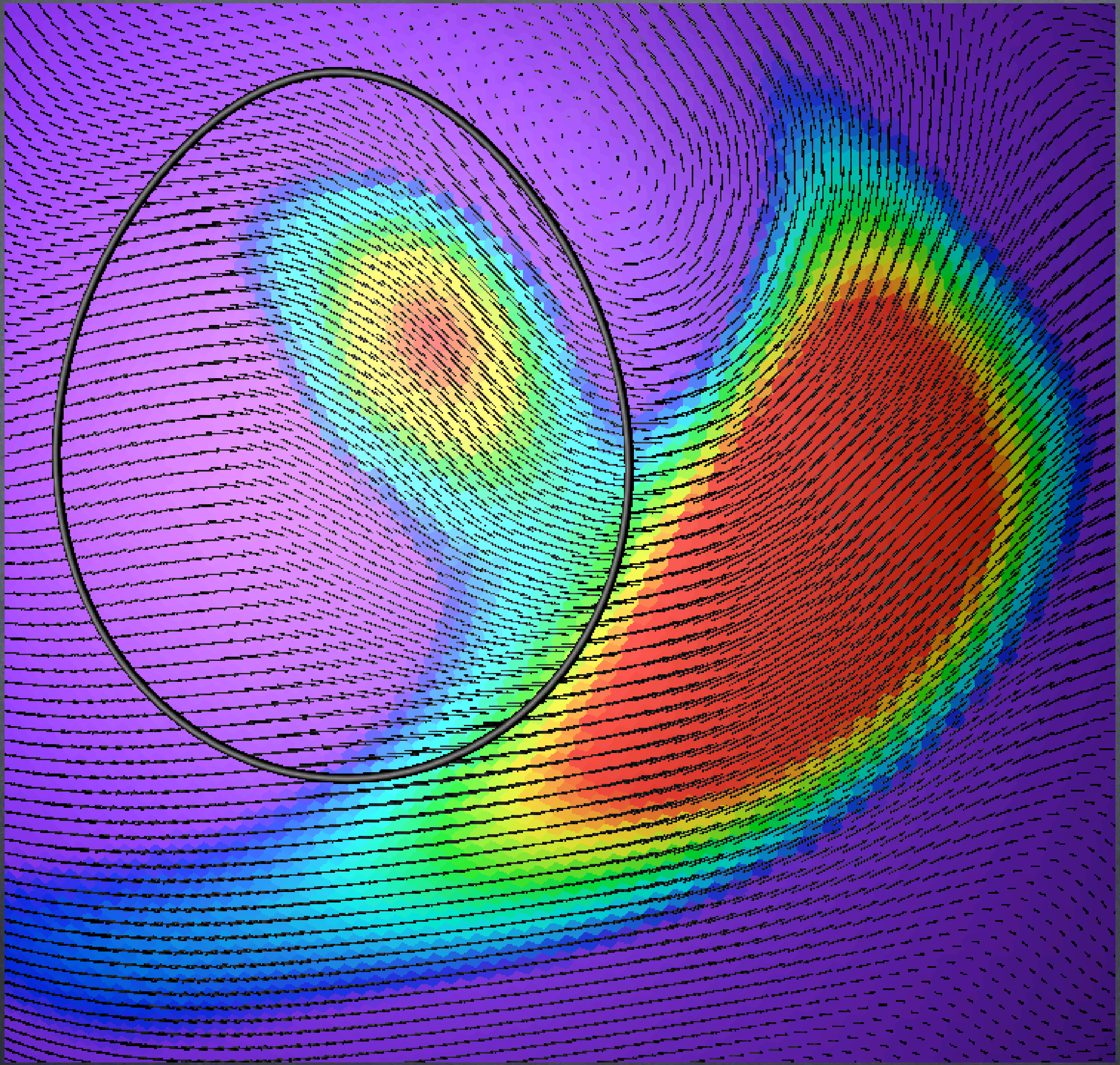
Look at shallow-water test case 5.

Geostrophically-balance flow confronts a 2 km mountain at t=0.

# Relative Vorticity, day 15

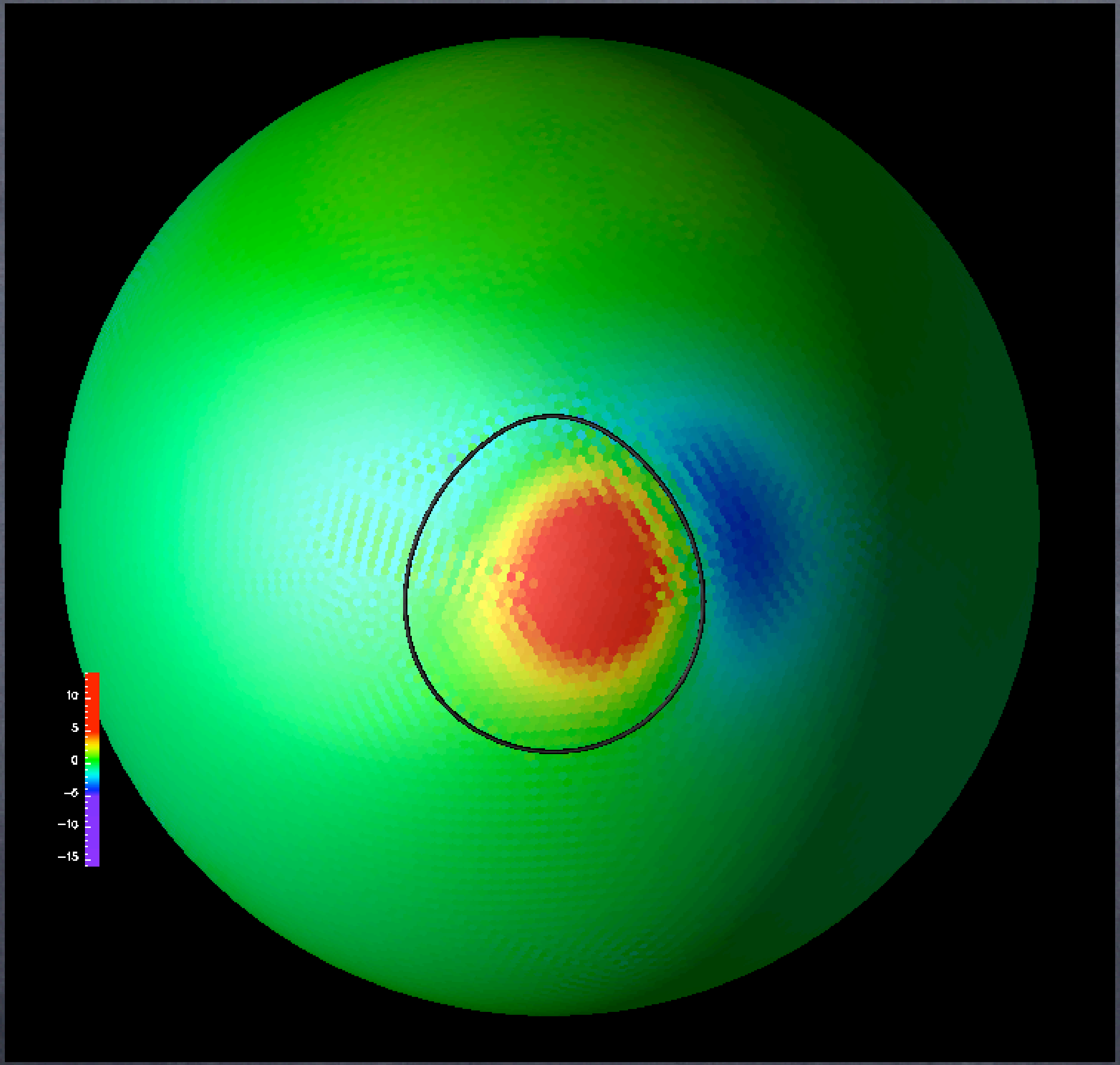


# Kinetic Energy, day=15



# Error in thickness field at t=24 hours

(ref solution is T511 global spectral model)



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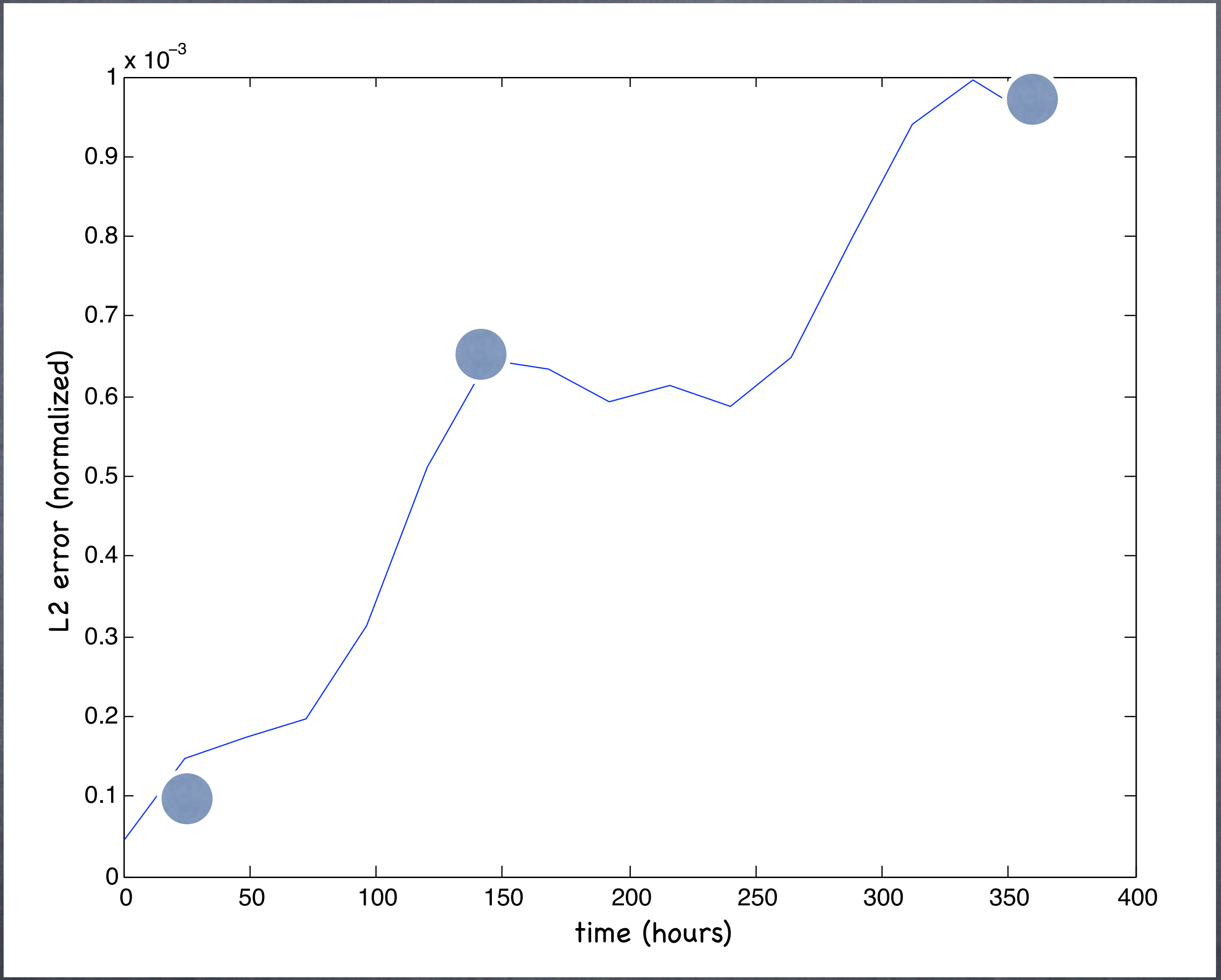
# L2 norm, SWTC#5 (40962, glevel 6)

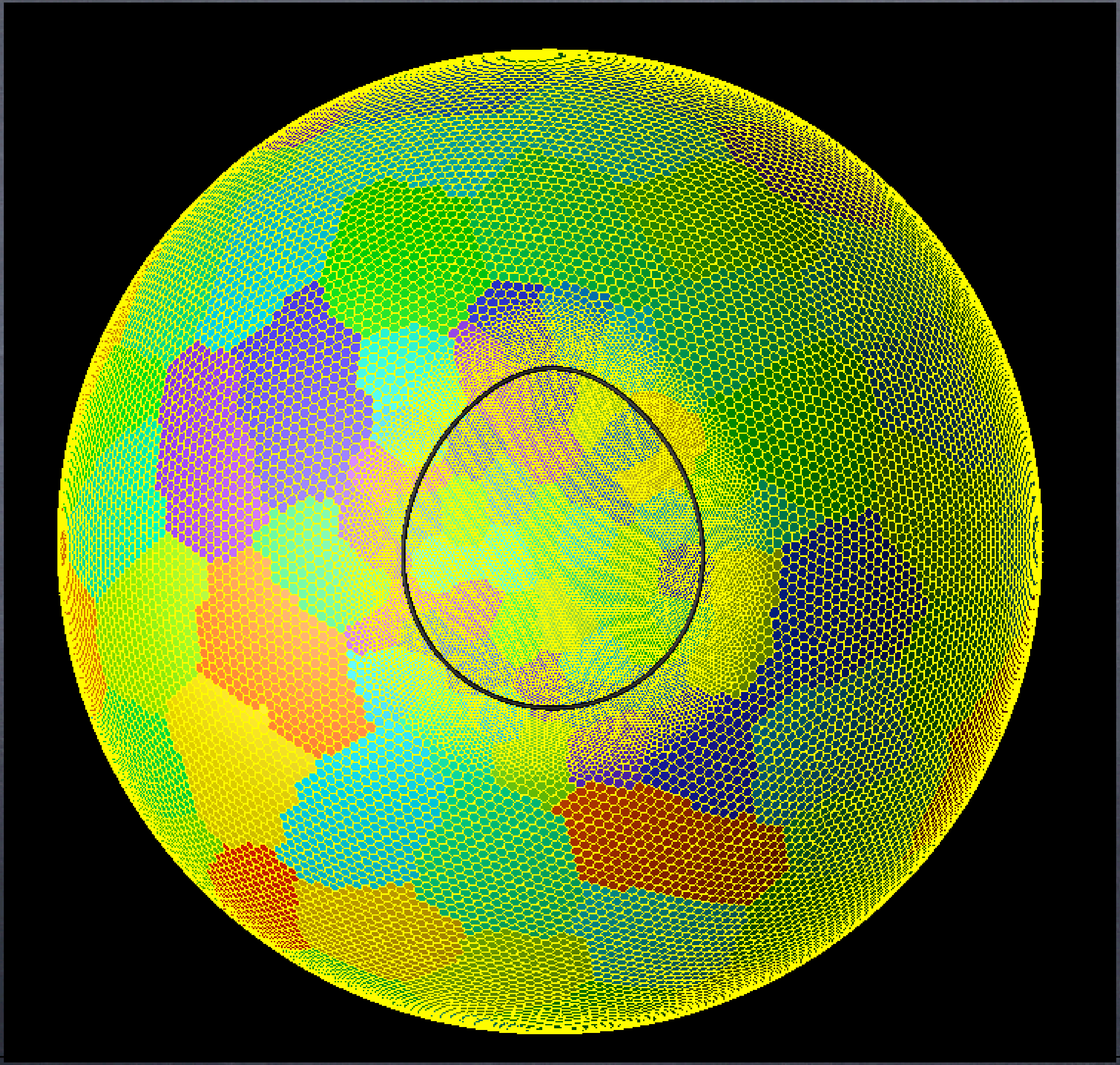



Tomita et al 2001

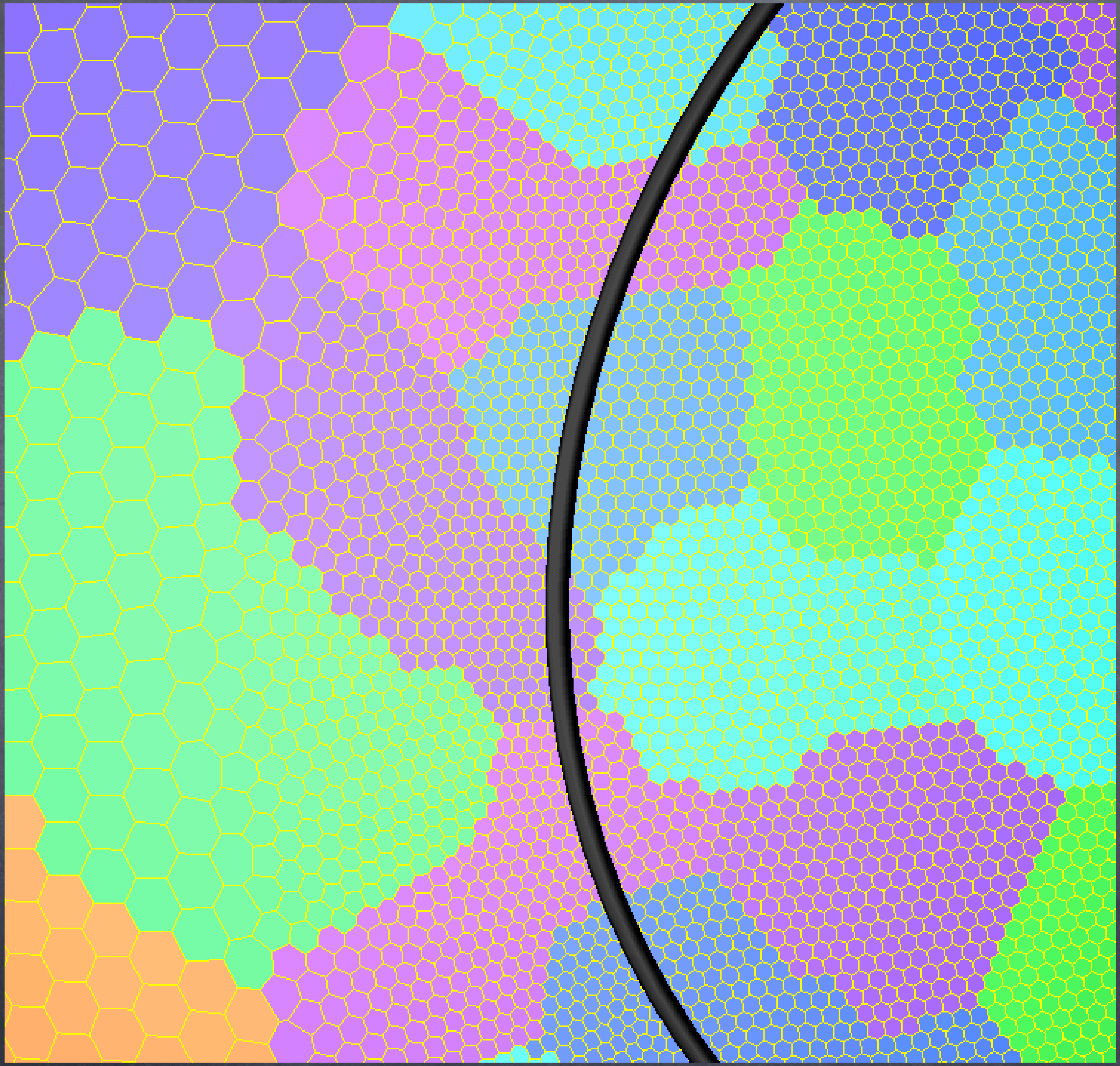
This all looks promising to this point.

We used a uniform SCVT in the previous example, but the model is applicable to any SCVT.

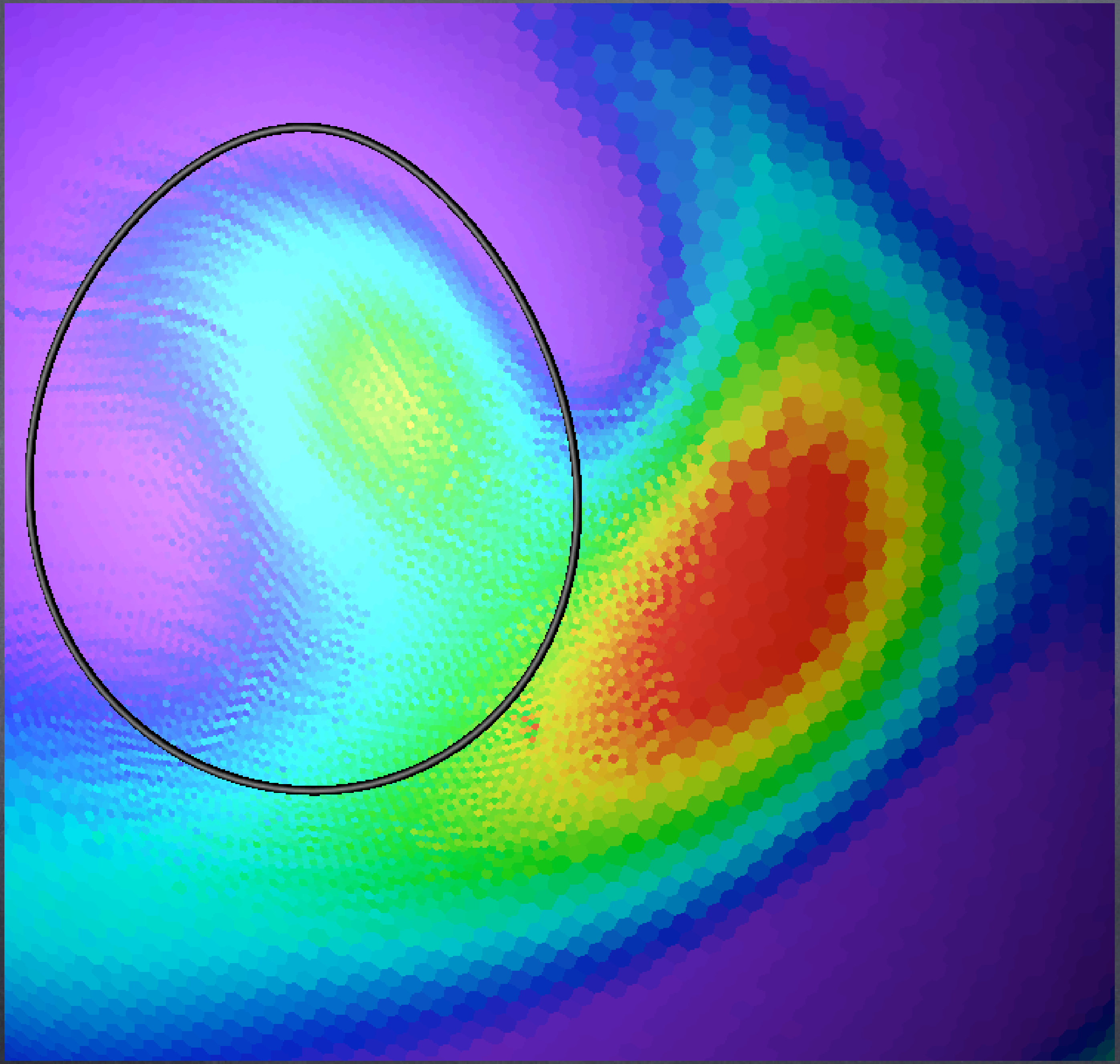
In SWTC#5 we know that the solution is dependent almost entirely on the accuracy of the forcing. We should be able to move our degrees of freedom to that region and get a more accurate solution.



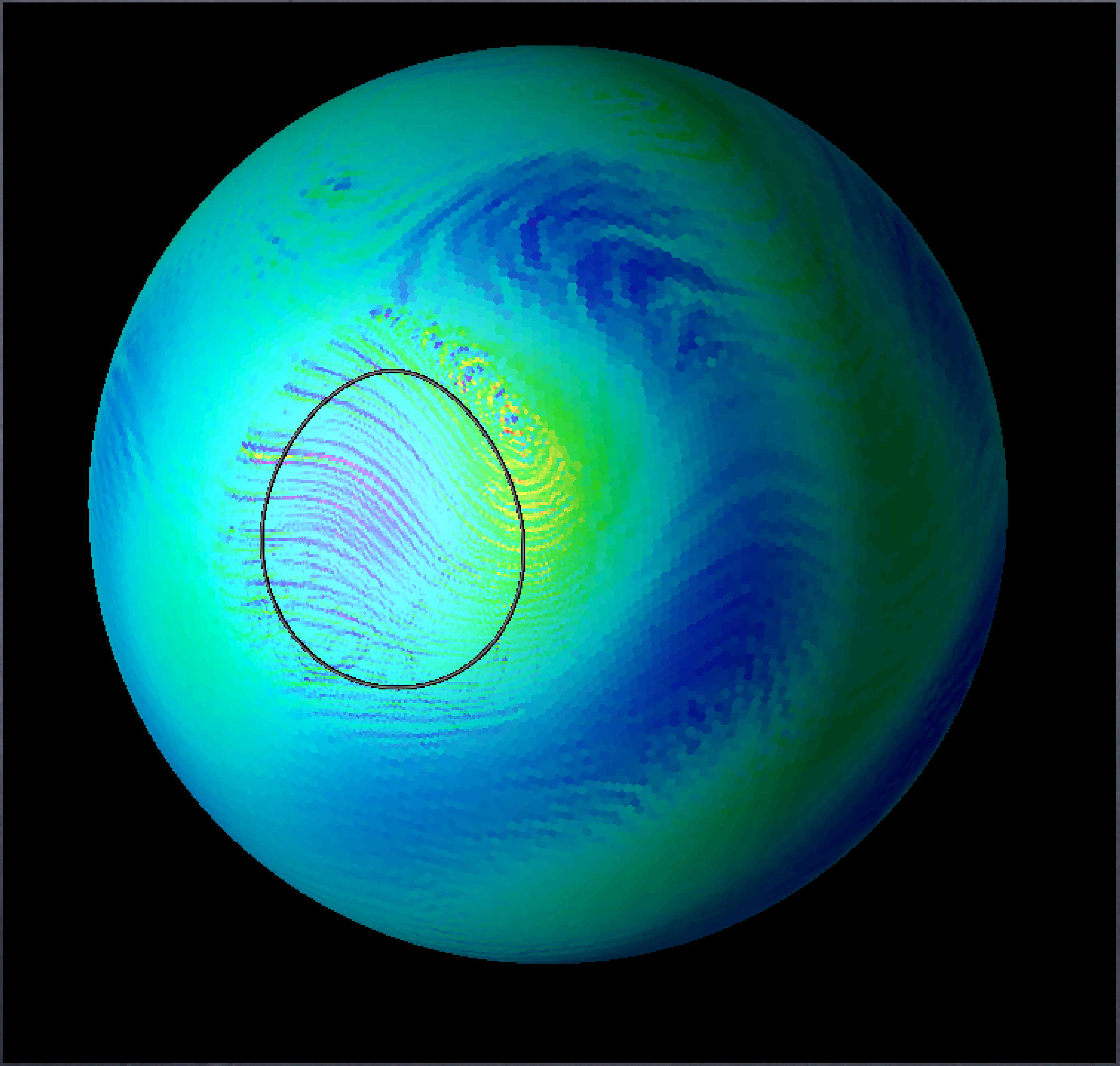
# A closer look at the variable resolution.



# Kinetic Energy, day 15

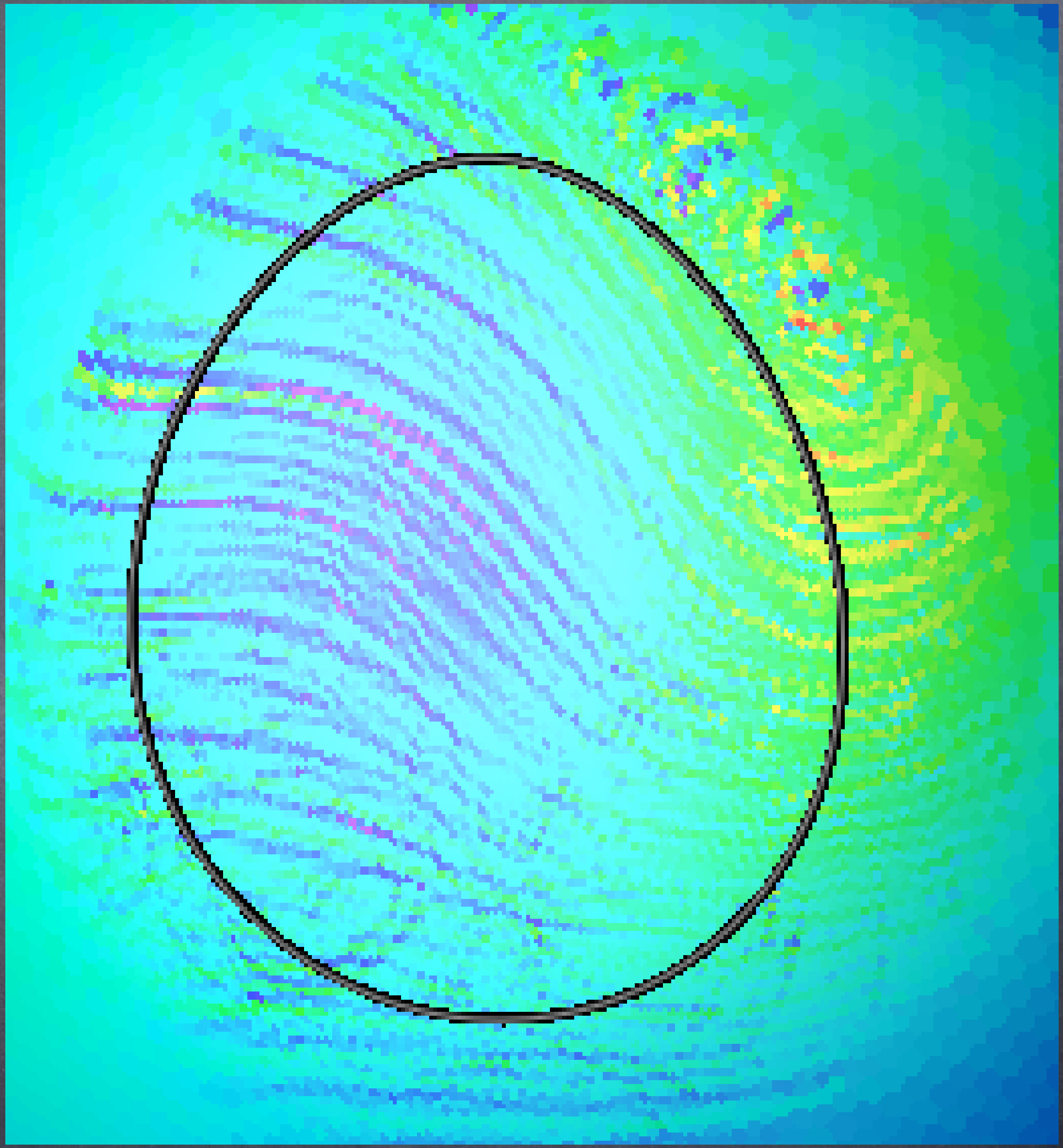


# Relative Vorticity, day 15.

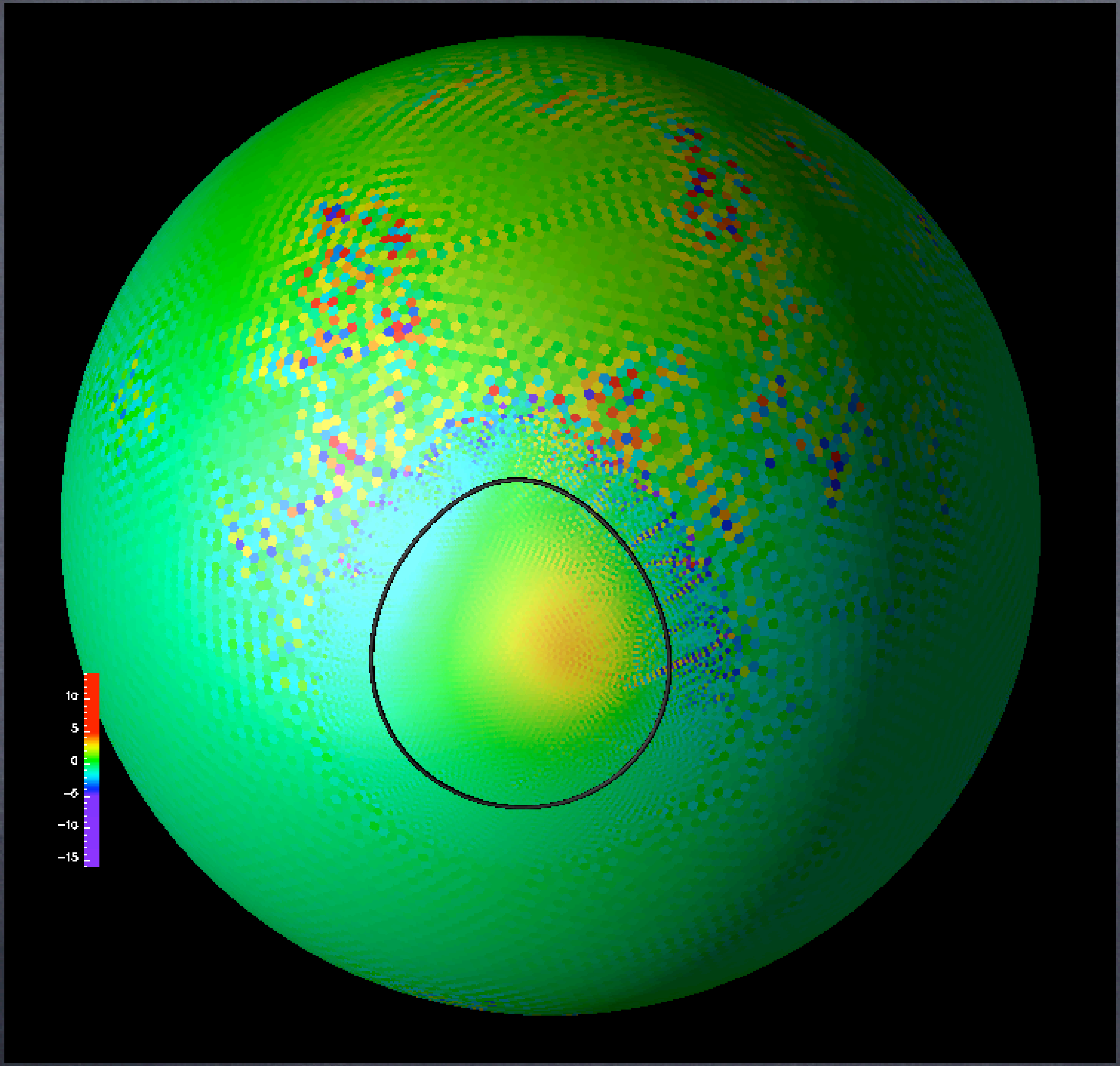


# A closer look around the mountain

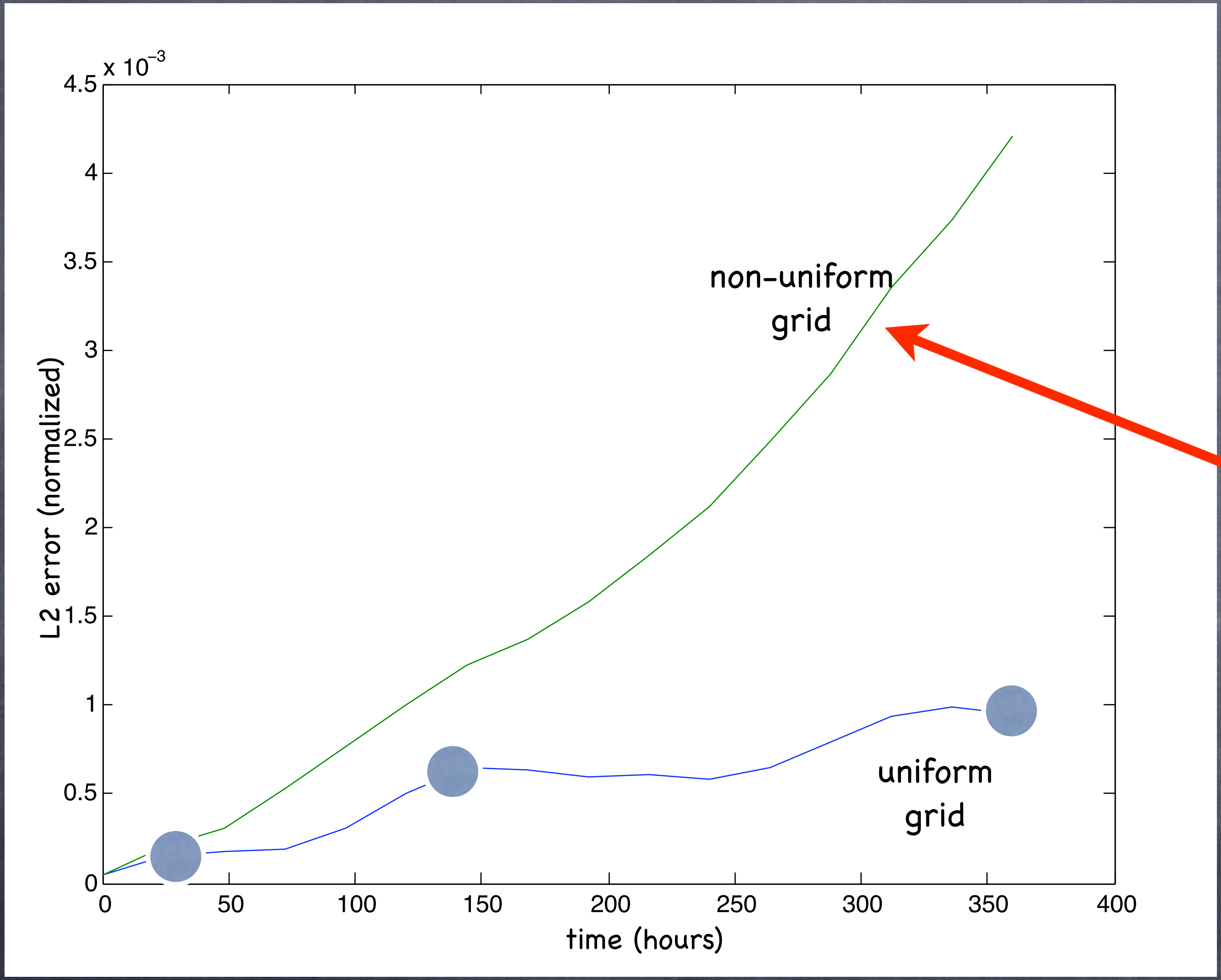
Striping is associated with nonlinear instability (potential enstropy in this case). Also, the dispersion relation indicated that this collocated method is susceptible to striping.



# Thickness error, t=24 hours



# Solution error as a function of time.




This has all of the characteristics of nonlinear instability.

# Summary of collocated grid findings.

Produces remarkably smooth flows when the grid is uniform. I attribute this to the lack of a checkerboard pattern for the Voronoi tessellation.

When the grid has variable resolution, solution becomes noisy (striping). The associated nonlinear stability is due to an unconstrained growth in potential enstrophy.

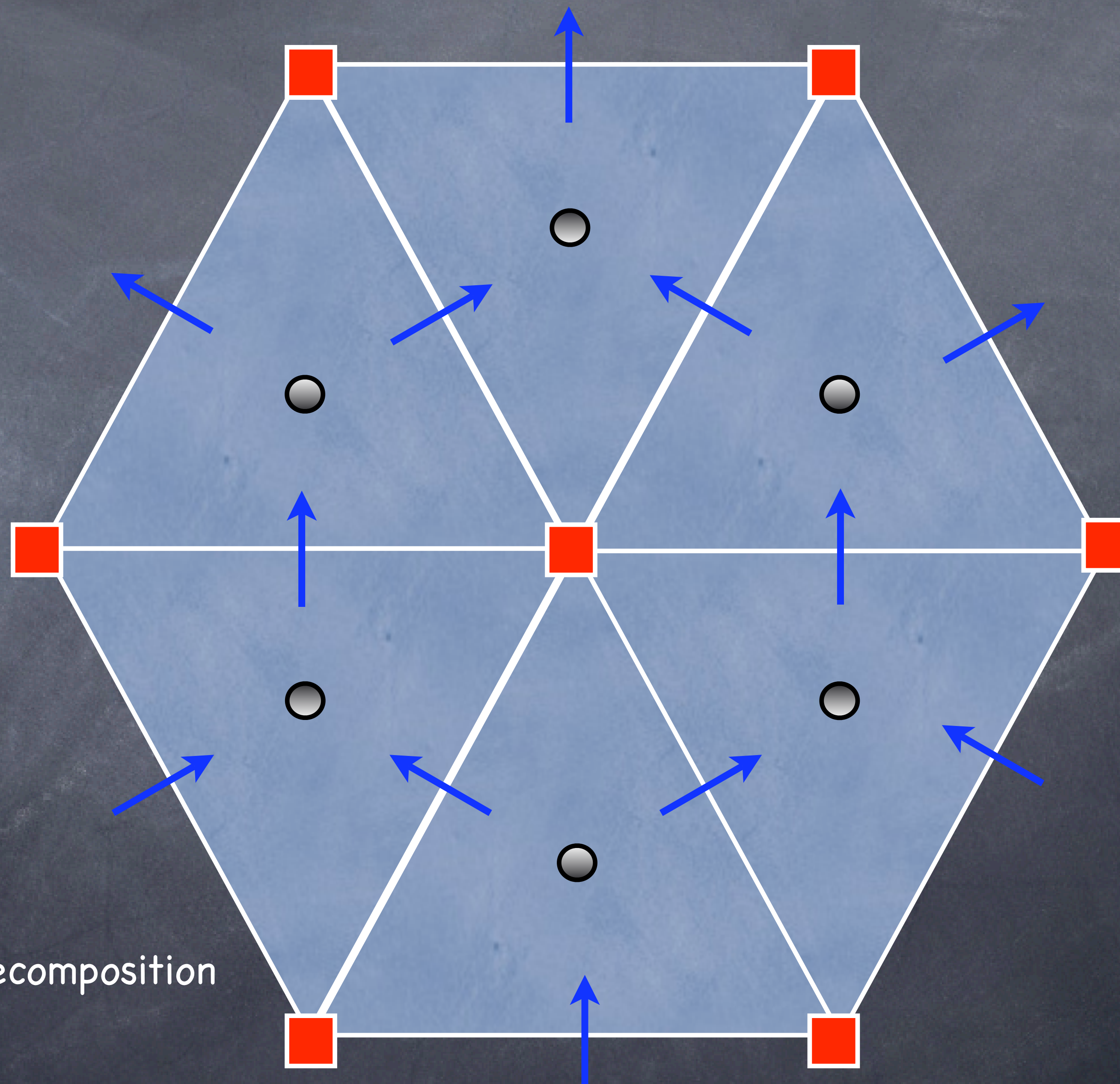
The lack of access to potential vorticity (and therefore) potential enstrophy compels me to try another method.

# Turning now to the triangular C-grid.

mass

momentum

vorticity



mass lives on the Delaunay triangulation and vorticity lives on the Voronoi diagram.

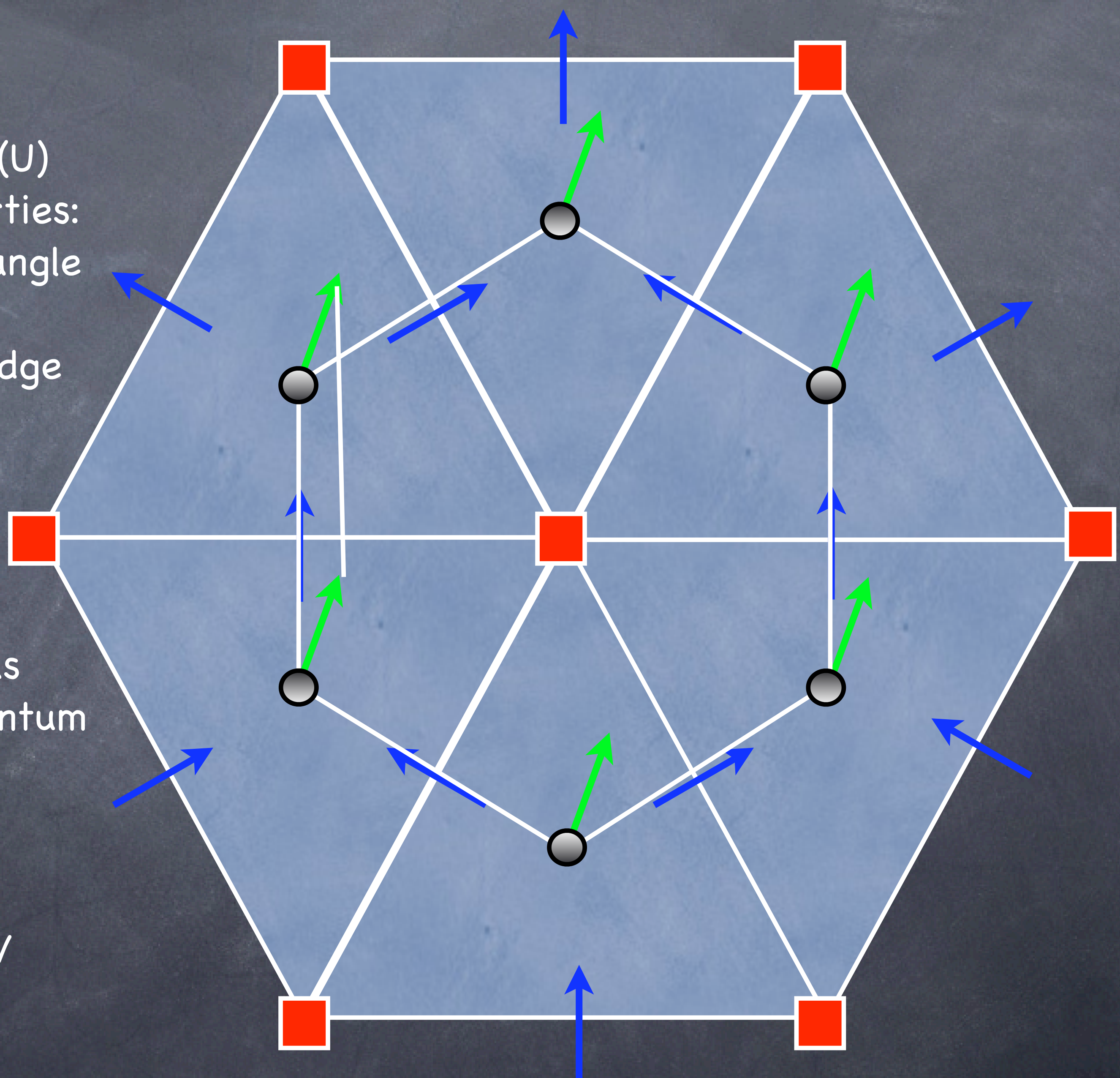
Attributes:

- PV is conserved.
- Unique discrete Helmoltz decomposition
- geostrophic balance

# Raviat-Thomas Element

Given the normal velocity components on the cell edges, there is a UNIQUE full velocity (U) vector with the following properties:

1. defined at the center of triangle
2. varies linearly over triangle
3. U dot n = velocity at each edge
4. div (U) equals divergence



PV flux across each edge of the Voronoi diagram is used as the eta k x u term in the momentum equation.

Guarantees that the momentum evolves with an accompanying PV field with bounded enstrophy!

# We are currently implementing this method

Positives:

Vorticity dynamics

Negatives:

Checkerboard mode in mass (monotone advection)

# Summary: #1 of 3

Over the last decade, an increased in accuracy due to running our quasi-uniform simulations at higher and higher resolution has somewhat masked the importance of having energy and potentially-enstrophy bounded schemes.

As we move to variable resolution, nested methods, or adaptive methods the importance of boundedness will likely reemerge.

This is because in regions of variable resolution, the truncation error is relatively large and, more importantly, not smooth.

# Summary: #2 of 3

Regardless of the physical system (atmosphere, ocean, land ice, etc.) there are two reasons to implement multi-scale methods.

#1. To allow the simulation of NEW processes not otherwise permitted in the modeling framework (i.e ocean eddies, hurricanes, etc).

#2. To allow for a more accurate solution. In that to expect a reduction in both truncation and solution error to a give PDE.

**#2 does not follow from #1.**

# Summary: #3 of 3

Can we obtain more accurate simulations using variable and/or adaptive grids than we can obtain with the commensurate quasi-uniform grid?

Specifics:
#1. Use the full shallow-water equations. Something like SWTC#5 or similar.

#2. Compare answers the results when using the same number of degrees of freedom positioned in a quasi-uniform manner.

What does it mean if we can not consistently get more accurate simulations using variable and or adaptive grids as compared to their uniform counterparts?


Thank you